REAL ANALYSIS LECTURE WEEK 1

WENYUAN LI

1. Lecture 1: History and Preliminaries

Calculus was systematically developed by Newton and Leibniz around the 1660s. Nowadays, we often attribute the fundamental theorem of calculus to Newton and Leibniz, which states that for a (continuously) differentiable function $f(x)$,

$$f(b) - f(a) = \int_a^b f'(x)dx.$$

Newton also introduced the infinite binomial series, which was generalized soon by Taylor into the infinite power series expansion of functions, now known as the Taylor expansion. Much of the results we learned in calculus appeared in the late 17th century and early 18th century, including for example L'Hopital's rule (due to Bernoulli) and Lagrange's mean value theorem that for a differentiable function $f(x)$, there exists $\xi \in (a, b)$, such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

More systematic discussion of continuous functions and limits, however, came much later. Cauchy, in the 1810s, made the attempts toward the modern definition of continuous functions. He also gave the first proof of Lagrange's mean value theorem (to some extent), that for a (continuously) differentiable function $f(x)$, there exists $\xi \in (a, b)$, such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Cauchy's definition of continuous functions was still verbal, but the arguments he used in the proofs were already very close to the modern definition of limits. Finally, in the 1860s, Weierstrass gave the modern definition of limits and continuous functions, who was the first to introduce the notation

$$\lim_{x \to a} f(x).$$

During the 1810s, Bolzano also independently gave a modern definition of continuous functions. He in particular gave the first proof of the intermediate value theorem in 1817, which says that for a continuous function $f(x)$ on an interval, if $f(a) < c$ and $f(b) > c$, then there exists $x \in [a, b]$ such that $f(x) = c$.

The theory of integration, now known as Riemann integration, also took a long time to develop. Riemann, in the 1850s, gave the definition of integral as Riemann sums. The way we will introduce Riemann integral in this course was developed by Darboux in the 1870s.

It was also in the 1860s that people first wrote down a rigorous definition of real numbers. Cantor and Dedekind gave two different but equivalent definitions of real numbers. Since the ancient Greek time, people have realized that not all quantities are conmeasurable, meaning that their ratio is a rational number. Nevertheless, Eudoxus, in the 300s BC, developed the theory of ratios by exhaustions, whose works were recorded in Euclid's elements. That was a prototype of Dedekind's theory of real numbers.

Cantor was also the person who developed set theory. He gave the first proof that rational numbers are countable while real numbers were not. This has become the foundations of almost all modern mathematics.

Finally, at the end of the 19th century, we realized that many analytic properties of the real numbers can be generalized to what is called metric spaces. This notion is introduced by Fréchet and Hausdorff in the 1900s, we will follow their path and generalize many of the results in analysis later from real numbers to metric spaces.

Why do people develop more and more complicated theories in the past few centuries even though most of calculus we know have already been developed in the 17th century without rigorous definitions?

Perhaps one reason is to respond to the questions and complaints on calculus, in particular, on infinitesimals (infinitely small numbers). For instance, Bolzano, being a philosopher himself, was aimed at setting up a new and rigorous foundation on calculus to avoid using intuitions from motions and geometry as Newton and Leibniz have been doing.

However, another important reason was perhaps the following. in the 18th and 19th century, people started to work with more and more complicated functions, where is became more and more necessary to lay down a more analytic solid foundation on the theory. For instance, Cauchy's treatment of limits and continuations showed up in the attempts of proving more complicated theorems.

In the 1820s, Fourier used infinite trigonometric series

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

to solve the heat equation (this led to the subject that is now called Fourier analysis). People started to encounter with discontinuous functions as the sum of the infinite trigonometric series, where a lot of intuitions for smooth functions start to break down. Riemann's definition of Riemann integrals appeared in his work on trigonometric series, and actually Cantor's definition of real numbers as limits also appeared in his studies on zero sets of trigonometric series.

In his lecture notes, Riemann wrote down without proof the first function which is continuous but nowhere differentiable, and Weierstrass constructed continuous but nowhere differentiable functions and gave the first proof. This lead to the more delicate discussions on limits, continuity, differentiability, and integrability of functions that are not power series.

As we will see in this course, these developments to some extent largely changed the shape of calculus, and formed the modern foundation of what we now call analysis.

2. Lecture 2: Limits of Sequences

Recall that a function (or mapping) is defined as follows:

Definition 2.1. *Let A, B be some sets. Suppose that for any $a \in A$, there exists an object associated to a in B, which we denote by $f(a) \in B$. Then we say that f is a function or a mapping from A to B.*

More formally, the data of a function is equivalent to the data of the set of pairs $(a, f(a))$ for all $a \in A$. Consider the set of pairs

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Then a function f from A to B is a subset $\Gamma_f \subset A \times B$, such that for any $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in \Gamma_f$, and we denote $b = f(a)$.

Definition 2.2. *A sequence is a function a from $\mathbb{N}$ to $\mathbb{R}$. We denote $a_n = a(n)$ and write the sequence in the form $\{a_n\}_{n\in\mathbb{N}}$.*

Given the definition, we can discuss the limit of a sequence. What does it mean when we say that

$$\lim_{n\to\infty} a_n = a?$$

Informally, this means that as n tends to infinity, a_n becomes infinitely close to a, or the difference of a_n and a becomes infinitely small. B. Bolzano in 1817 first tried to make this idea rigorous:

Definition 2.3 (Bolzano, 1817, informal)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Then we say that a_n converges to a if the difference a_n and a can be made smaller than any quantity provided n can be taken as large as we want.*

In the above definition, we interpret limit of a sequence by requiring the difference of a_n and a to be smaller than any quantity. Of course, if we ask the difference of a_n and a to be less than ϵ for instance, this may not happen in the first few terms, however, we hope that it could happen if we make n very large.

A. L. Cauchy in his 1820 book was using infinitely small quantities in his definitions. However, he interpreted limits via inequalities involving ϵ: that for an infinitesimal number $\epsilon > 0$, we can let N be a number such that when $n \geq N$, a_n will be greater than $a - \epsilon$ and less than $a + \epsilon$. Here, ϵ stands for the error.

Cauchy did not carefully use logic quantifiers. However, accoring to our above interpretation, we see that what we should expect is the following: we can consider any possible requirement ϵ on the error (between a_n and a). Once a requirement ϵ on the error is given to us, we will try to find very large n such that the requirement is satisfied. This is to say, we want to ensure that there exists a range $N \leq n < \infty$ where the requirement is satisfied.

Finally, Weierstrass in his 1865 lecture introduced the following definition of limits, which we use nowadays:

Definition 2.4 (Weierstrass, 1865)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. We say that a_n converges to a if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $n \geq N$, we have $|a_n - a| \leq \epsilon$. If there are no such a that satisfies the requirement, then we say that a_n diverges.*

Remark 2.1. *Based on the above definition, we say that a_n does not converge to a if there exists $\epsilon > 0$, such that for any $N \in \mathbb{N}$, there exists $n \geq N$, we have $|a_n - a| > \epsilon$.*

3. Lecture 3: Properties and Examples

We summarize some important properties of limits of sequences. The proofs show how one could use the condition that a sequence converges to a: one choose an error ϵ, and then obtain inequalities based on the error that is chosen.

Definition 3.1. *A set $A \subset \mathbb{R}$ is bounded if there exists $M \in \mathbb{R}$, such that $A \subset [-M, M]$. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded if there exists $M \in \mathbb{R}$, such that for any $n \in \mathbb{N}$, $a_n \subset [-M, M]$.*

Theorem 3.1. *(a) $\{a_n\}_{n\in\mathbb{N}}$ converges to a if and only if any neighborhood $[a-\epsilon, a+\epsilon]$ of a contains all but finitely many terms in $\{a_n\}_{n\in\mathbb{N}}$. (b) If $\{a_n\}_{n\in\mathbb{N}}$ converges to a and a', then $a = a'$. (c) If $\{a_n\}_{n\in\mathbb{N}}$ converges, then $\{a_n\}_{n\in\mathbb{N}}$ is bounded.*

Proof. (a) Suppose $\{a_n\}_{n\in\mathbb{N}}$ converges to a. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$|a_n - a| \leq \epsilon,$$

which means for any $n \geq N$, $a_n \in [a - \epsilon, a + \epsilon]$.

Conversely, suppose for any $[a-\epsilon, a+\epsilon]$, it contains all but finitely many terms in $\{a_n\}_{n\in\mathbb{N}}$. We may assume that the largest natural number such that a_n is not contained in $[a-\epsilon, a+\epsilon]$ is equal to N. Then, for any $n > N$, we know $a_n \in [a-\epsilon, a+\epsilon]$, which means that

$$|a_n - a| \leq \epsilon.$$

(b) We prove by contrapositive. Suppose that $a > a'$. Let $\epsilon = (a-a')/3$. Then since $a_n \to a$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$|a_n - a| \leq \epsilon,$$

which means that $a_n \in [a-\epsilon, a+\epsilon]$. Since $a_n \to a'$, there also exists $N' \in \mathbb{N}$ such that for any $n \geq N$,

$$|a_n - a'| \leq \epsilon,$$

which means that $a_n \in [a'-\epsilon, a'+\epsilon]$. This means that when $n \geq \max\{N, N'\}$, we have

$$a_n \in [a'-\epsilon, a'+\epsilon] \text{ and } a_n \in [a-\epsilon, a+\epsilon].$$

When $\epsilon = (a-a')/3$, $[a'-\epsilon, a'+\epsilon] \cap [a'-\epsilon, a'+\epsilon] = \varnothing$. This leads to a contradiction.

(c) Let $\epsilon = 1$. Then since $a_n \to a$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$|a_n - a| \leq 1,$$

which means $a_n \in [a-1, a+1]$. Then consider

$$M = \max\{|a_1|, \ldots, |a_N|, |a-1|, |a+1|\}.$$

We know that for any $n \in \mathbb{N}$, $|a_n| \leq M$. □

Some basic properties of limits of sequences under additions and multiplications. The following statement is useful: $|x \pm y| \leq |x| + |y|$.

Theorem 3.2. *Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$.*
(a) $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$.
(b) $\lim_{n\to\infty} a_n b_n = ab$.
(c) $\lim_{n\to\infty} 1/a_n = 1/a$ if $a_n \neq 0$ and $a \neq 0$.

Proof. (a) Since $a_n \to a$ and $b_n \to b$, for any $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$,

$$|a_n - a| \leq \epsilon/2.$$

and there also exists $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$,

$$|b_n - b| \leq \epsilon/2.$$

Therefore, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ with $N = \max\{N_1, N_2\}$, for any $n \geq N$,

$$|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| \leq \epsilon.$$

(b) First, since $a_n \to a$ and $b_n \to b$, we know that they are bounded. There exists $M_1 \in \mathbb{R}$ and $M_2 \in \mathbb{R}$ such that for any $n \in \mathbb{N}$,

$$|a_n| \leq M_1, \quad |b_n| \leq M_2.$$

Take $M = \max\{a, b, M_1, M_2\}$. Then for any $n \in \mathbb{N}$,

$$|a_n| \leq M, \quad |b_n| \leq M, \quad |a| \leq M, \quad |b| \leq M.$$

Since $a_n \to a$ and $b_n \to b$, for any $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$,

$$|a_n - a| \leq \epsilon/2M.$$

and there also exists $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$,

$$|b_n - b| \leq \epsilon/2M.$$

Therefore, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ with $N = \max\{N_1, N_2\}$, for any $n \geq N$,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \leq |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n||b_n - b| + |a_n - a||b| \leq \epsilon. \end{aligned}$$

This completes the proof. □

Here are some examples of limits of sequences. These examples show how to show that a sequence has a given limit: one needs to start from any error ϵ, and provide a general formula for the natural number $N \in \mathbb{N}$ such that the requirement (in the definition of limits) holds.

In the definitions of limits, we often need to find a large natural number $N \in \mathbb{N}$. The following property is often used for finding large integers:

Proposition 3.3 (Archimedean principle)**.** *For any real number $\alpha \in \mathbb{R}$, there exists a natural number $N \in \mathbb{N}$ such that $N \geq \alpha$.*

We will only be able to give a proof of the above proposition after systematically introducing the theory on real numbers. For now we will simply assume the above (seemly very obvious) proposition.

We also recall the binomial theorem, which says that

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + b^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \dots.$$

Proposition 3.4. *(a) For any $p > 0$, $\lim_{n\to\infty} \dfrac{1}{n^p} = 0$.*

(b) For any $p > 0$, $\alpha \in \mathbb{R}$, $\lim_{n\to\infty} \dfrac{n^\alpha}{(1+p)^n} = 0$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. (a) For any $\epsilon > 0$, there exists $N \geq (1/\epsilon)^{1/p}$ by the Archimedean principle. Then for any $n \geq N$,

$$0 < \frac{1}{n^p} \leq \frac{1}{N^p} \leq \epsilon.$$

(c) For any $\epsilon > 0$, we want to find $N \in \mathbb{N}$ such that for any $n \geq N$, we have

$$0 \leq \sqrt[n]{n} - 1 \leq \epsilon.$$

This means we want to obtain that $\sqrt[n]{n} \leq 1 + \epsilon$ or equivalently $n \leq (1+\epsilon)^n$. Using the binomial theorem, we have

$$(1+\epsilon)^n \geq \frac{n(n-1)}{2}\epsilon.$$

Therefore, for any $\epsilon > 0$, there exists $N \geq 2/\epsilon + 1$ by the Archimedean principle. Then for any $n \geq N$,

$$(1+\epsilon)^n \geq (1+\epsilon)^N \geq \frac{N(N-1)}{2}\epsilon \geq N.$$

This means that

$$0 \leq \sqrt[n]{n} - 1 \leq \epsilon.$$

□

4. Lecture 4: Cauchy Sequences and Subsequences

We established the theory of convergence of sequences in the previous lectures. Now a sequence $\{a_n\}_{n\in\mathbb{N}}$ converges to a has a formal definition. However, such a theory has at least one disadvantage: we can only show that a sequence $\{a_n\}_{n\in\mathbb{N}}$ converges when we know the limit a. If we cannot tell what the potential limit of $\{a_n\}_{n\in\mathbb{N}}$ is, there is of course no way to compute the error $|a_n - a|$ as the potential limit a is unknown.

In this lecture, we will try to establish some theory of convergence without knowing the limit. Instead of computing the error between a_n and the limit a, we compute the error between a_n and a_m.

Definition 4.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Then we say $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $m, n \geq N$, we have*

$$|a_n - a_m| \leq \epsilon.$$

Theorem 4.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Then $\{a_n\}_{n\in\mathbb{N}}$ is convergent if and only if it is a Cauchy sequence.*

Proof. We prove that if $\{a_n\}_{n\in\mathbb{N}}$ is convergent, then it is a Cauchy sequence. Suppose that $a_n \to a$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $n \geq N$,

$$|a_n - a| \leq \epsilon/2.$$

This means that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $m, n \geq N$,

$$|a_n - a_m| = |(a_n - a) + (a - a_m)| \leq |a_n - a| + |a_m - a| \leq \epsilon.$$

which shows that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. □

Remark 4.1. *Note that the following condition is not equivalent to the Cauchy condition does not imply convergence of a sequence: for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $n \geq N$, we have*

$$|a_n - a_{n+1}| \leq \epsilon.$$

The other direction of the theorem is harder and in fact requires the theory of real numbers. Therefore, we will go back to it later after discussing real numbers.

One particular useful application is the following convergence result:

Definition 4.2. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Then we say $\{a_n\}_{n\in\mathbb{N}}$ is monotonely increasing (resp. decreasing) if for any $n \in \mathbb{N}$, $a_n \leq a_{n+1}$ (resp. $a_n \geq a_{n+1}$). We say it is monotone if it is monotonely increasing or decreasing.*

Theorem 4.2. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a monotone sequence that is bounded. Then $\{a_n\}_{n\in\mathbb{N}}$ is convergent.*

Proof. We show by contrapositive. Assume that $\{a_n\}_{n\in\mathbb{N}}$ diverges and is monotonely increasing, we show that for any $M \in \mathbb{R}$ (possible bound), there exists $n \in \mathbb{N}$ (that we will find later), such that

$$|a_n| \geq M.$$

Since $\{a_n\}_{n\in\mathbb{N}}$ diverges, there exists $\epsilon > 0$ such that for any $N \in \mathbb{N}$, there exists $n \geq m \geq N$ such that (we can take out the absolute value because the sequence is increasing)

$$a_n - a_m = |a_n - a_m| > \epsilon.$$

Let $N = 1$, then there exists $n_2 \geq n_1 \geq 1$ such that

$$a_{n_2} - a_{n_1} \geq \epsilon.$$

Let $N = n_2$, then there exists $n_4 \geq n_3 \geq n_2$ such that

$$a_{n_4} - a_{n_3} \geq \epsilon.$$

Let $N = n_{2k}$, then there exists $n_{2k+2} \geq n_{2k+1} \geq n_{2k}$ such that

$$a_{n_{2k+2}} - a_{n_{2k+1}} \geq \epsilon.$$

This implies that

$$\begin{aligned} a_{n_{2k+2}} - a_1 &= (a_{n_{2k+2}} - a_{n_{2k+1}}) + (a_{n_{2k+1}} - a_{n_{2k}}) + \cdots + (a_{n_2} - a_{n_1}) + (a_{n_1} - a_1) \\ &> \epsilon + 0 + \epsilon + 0 + \cdots + \epsilon + 0 = (k+1)\epsilon. \end{aligned}$$

Now choose $k \geq (M + a_1)/\epsilon - 1$ by the Archimedean property and choose $n = n_{2k+2}$. Then we know that

$$a_n \geq (k+1)\epsilon + a_1 \geq M.$$

Since $M \in \mathbb{R}$ can be any real number, this implies that $\{a_n\}_{n\in\mathbb{N}}$ is not bounded. □

The above theorem can be deduced using Cauchy's convergence theorem. However, the proof of the theorem is delicate, so we will postpone the proof after discussing real numbers.

We have seen in the previous lectures that a convergent sequence is also bounded, and a monotone bounded sequence is also convergent. However, not all bounded sequences are convergent (if one drops the monotone condition): Consider the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$. Then the odd terms are equal to -1 while the even terms are equal to 1, and one can show that the sequence does not converge.

However, if we consider only the odd (resp. even) terms, then the sequence we get is constantly -1 (resp. 1). We can make the idea formal through the following definition:

Definition 4.3. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Consider an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ where for any $k \in \mathbb{N}$, $n_k < n_{k+1}$. Then the sequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ is called a subsequence of $\{a_n\}_{n\in\mathbb{N}}$.*

Theorem 4.3. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Then $\{a_n\}_{n\in\mathbb{N}}$ converges to a if and only if any subsequence of $\{a_n\}_{n\in\mathbb{N}}$ converges to a.*

Going back to the original question, it is actually the case that any bounded sequence, though not necessarily convergent, has a convergent subsequence:

Theorem 4.4 (Bolzano 1817; Weierstrass, 1865)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence. Then there exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ that is convergent.*

The above theorem can be deduced using Cauchy's convergence theorem. However, the proof of the theorem is delicate, so we will postpone the proof after discussing real numbers.

5. Lecture 5: Series (of Nonnegative Terms)

One special type of sequences is infinite series. We will now recall results on infinite series that we learned in calculus and prove these results rigorously.

Definition 5.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. We write*

$$\sum_{k=p}^{q} a_k = a_p + a_{p+1} + \cdots + a_{q-1} + a_q.$$

In particular, we can define an associated sequence of partial sums

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

We say the associated sequence of partial sums $\{s_n\}_{n\in\mathbb{N}}$ *is an infinite series, and write*

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_n = \lim_{n\to\infty} s_n.$$

Suppose $\{s_n\}_{n\in\mathbb{N}}$ *converges to* s*, we say that the infinite series* $\sum_{n=1}^{\infty} a_n$ *converges to* s*.*

We can translate Cauchy's convergence criterion into the setting of infinite series. Since $s_m - s_{n-1} = \sum_{k=n}^{m} a_k$, we obtain that:

Theorem 5.1. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series. Then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if for any* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$*, such that for any* $m \geq n \geq N$*, we have*

$$\left| \sum_{k=n}^{m} a_k \right| < \epsilon.$$

When we take $m = n$ in the above theorem, we obtain a necessary condition for a series to converge:

Theorem 5.2. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series. If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$*.*

Remark 5.1. *This is not a sufficient condition. In other words, there exists a series where* $\lim_{n\to\infty} a_n = 0$ *but* $\sum_{n=1}^{\infty} a_n$ *diverges.*

We can also translate the theorem that monotone bounded sequences converge into the setting of infinite series. Since $s_n - s_{n-1} = a_n$, s_n is monotonely increasing if and only if for any $n \in \mathbb{N}$, $a_n \geq 0$, and thus we obtain that:

Theorem 5.3. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with nonnegative terms. Then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the partial sum is bounded.*

Given these basic results, we can now prove some basic convergence criteria for infinite series that we have seen in calculus:

Theorem 5.4 (Comparison test)**.** *(a) If there exists* $N_0 \in \mathbb{N}$ *such that for any* $n \geq N_0$*,* $0 \leq a_n \leq b_n$*, and* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *also converges.*

(b) If there exists $N_0 \in \mathbb{N}$ *such that for any* $n \geq N_0$*,* $a_n \geq b_n \geq 0$*, and and* $\sum_{n=1}^{\infty} b_n$ *diverges, then* $\sum_{n=1}^{\infty} a_n$ *also diverges.*

Proof. (a) Since $\sum_{n=1}^{\infty} b_n$ converges, we know that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $m \geq n \geq N$, we have

$$\sum_{k=n}^{m} b_k \leq \epsilon.$$

Therefore, for any $\epsilon > 0$, there exists $N' \in \mathbb{N}$ with $N' \geq \max\{N, N_0\}$, such that for any $m \geq n \geq N$, we have

$$\sum_{k=n}^{m} a_k \leq \sum_{k=n}^{m} b_k \leq \epsilon.$$

This implies that $\sum_{n=1}^{\infty} a_n$ converges. (b) is the contrapositive of (a). □

Finally, we recall the sum of geometric series:

Theorem 5.5. *The infinite series*

$$\sum_{n=0}^{\infty} x^n$$

converges if and only if $|x| < 1$*, and when it converges,*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

6. Lecture 6: Examples of Infinite Series

We define the Euler constant e. This will be an application of the monotone convergence criterion in the previous lectures. Recall that we define $0! = 1$, $n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$.

Theorem 6.1. *The infinite series*

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

converges, where we denote the limit by e*.*

Proof. We know that for any $n \geq 1$,

$$\frac{1}{n!} = \frac{1}{n(n-1)\cdots 2 \cdot 1} \leq \frac{1}{2 \cdot 2 \cdot \dots \cdot 2 \cdot 1} = \frac{1}{2^{n-1}}.$$

Therefore, for any $n \geq 1$, the partial sum is bounded

$$\sum_{k=0}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + 1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} \leq 1 + 1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} = 3.$$

Hence the series converges. □

Theorem 6.2.

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Proof. Let

$$t_n = \left(1 + \frac{1}{n}\right)^n.$$

Then by the binomial expansion, we know that

$$\begin{aligned} t_n &= 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \binom{n}{3}\frac{1}{n^3} + \dots + \binom{n}{n}\frac{1}{n^n} \\ &= 1 + \frac{n}{1!}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \dots + \frac{n!}{n!}\frac{1}{n^n} \\ &= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\dots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

This implies that for any $n \in \mathbb{N}$,

$$t_n \leq e.$$

On the other hand, we also know that for any $m \in \mathbb{N}$,

$$t_n \geq 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\dots\left(1 - \frac{m-1}{n}\right).$$

Since we know that $(1 - 1/n) \to 0, \dots, (1-1/n)(1-2/n)\dots(1-(m-1)/n) \to 0$ as $n \to \infty$, there exists $N_1, \dots N_{m-1} \in \mathbb{N}$, such that for any $n \geq N_1$,

$$\left(1 - \frac{1}{n}\right) \geq 1 - \frac{2!}{m-1}\frac{\epsilon}{2},$$

for any $n \geq N_2$,

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \geq 1-\frac{3!}{m-1}\frac{\epsilon}{2},$$

and for any $n \geq N_{m-1}$,

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \geq 1-\frac{m!}{m-1}\frac{\epsilon}{2}.$$

This implies that given $N = \max\{N_1, \ldots, N_{m-1}\}$, for any $n \geq N$, we have

$$t_n \geq 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{m!}-\frac{\epsilon}{2}.$$

Since this holds for any $m \in \mathbb{N}$, and $1+1/1!+1/2!+1/3!+\cdots+1/m! \to e$, we know that there exists $N' \in \mathbb{N}$, such that for any $m \geq N'$,

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{m!} \geq e-\frac{\epsilon}{2}.$$

Therefore, when $n \geq \max\{N_1, \ldots, N_{m-1}, N'\}$, we know that

$$t_n \geq 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{m!}-\frac{\epsilon}{2} \geq e-\epsilon.$$

We can conclude that

$$\lim_{n\to\infty} t_n = e.$$

□

We also discuss the convergence of the p-series:

Theorem 6.3. *For any $p > 1$, the following infinite series converges*

$$\sum_{n=0}^{\infty} \frac{1}{n^p}.$$

Proof. Let $\sigma_N = \sum_{n=1}^{N} 1/n^p$ be the partial sum. There exists $k \in \mathbb{N}$ such that $2^k \geq N$ (by the Archimedean principle). We show that

$$\sigma_{2^k} \leq 1+2^{(1-p)}+\cdots+2^{(1-p)(k-1)}.$$

In fact, this is because $\sigma_1 = 1$ and

$$\begin{aligned}\sigma_{2^k}-\sigma_{2^{k-1}} &= \frac{1}{(2^{k-1}+1)^p}+\frac{1}{(2^{k-1}+2)^p}+\cdots+\frac{1}{(2^k)^p} \\ &\leq \frac{1}{(2^{k-1})^p}+\frac{1}{(2^{k-1})^p}+\cdots+\frac{1}{(2^{k-1})^p} = 2^{(1-p)(k-1)}.\end{aligned}$$

Therefore, for any $N \in \mathbb{N}$, we can conclude that

$$\sigma_N \leq \sigma_{2^k} \leq 1+2^{(1-p)}+\cdots+2^{(1-p)(k-1)} < \frac{1}{1-2^{1-p}}.$$

□

7. Lecture 7: Ratio and Root Test, Power Series

Using comparison test, we can compare a series with a geometric series and deduce its convergence or divergence. Here, we prove some simple cases of the ratio and root test:

Theorem 7.1 (Root test). *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series.*
(a) Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges;
(b) Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. We only prove (a). Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \beta < 1$. Then for $\epsilon = (1-\beta)/2$, there exists $N \in \mathbb{N}$, such that for any $n \geq N$,

$$|\sqrt[n]{|a_n|} - \beta| \leq \epsilon, \quad |a_n| \leq (\beta+\epsilon)^n.$$

Therefore, by the comparison test, the series converges. □

Theorem 7.2 (Ratio test). *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series.*
(a) Suppose $\lim_{n\to\infty} |a_n|/|a_{n-1}| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges;
(b) Suppose $\lim_{n\to\infty} |a_n|/|a_{n-1}| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

The following theorem shows that the root test is always stronger than the ratio test, and when the limit exists, the ratio and root tests are equivalent to each other.

Theorem 7.3. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Then*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|a_n|}{|a_{n-1}|}.$$

when the right hand side exists.

Proof. Suppose $\lim_{n\to\infty} |a_n|/|a_{n-1}| = \beta$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that when $n \geq N$,

$$||a_n|/|a_{n-1}| - \beta| \leq \epsilon, \quad (\beta - 2\epsilon)^{n-N}|a_N| \leq |a_n| \leq (\beta+2\epsilon)^{n-N}|a_N|.$$

Take $\epsilon > 0$ such that

$$\frac{(\beta-2\epsilon)^{n-1}}{(\beta+2\epsilon)^{n-1}} \geq 1 - \epsilon.$$

Then there exists $N \in \mathbb{N}$, such that for any $n \geq N$,

$$\sqrt[n]{|a_N|(\beta-\epsilon)^{-N}} \cdot (\beta - \epsilon) \leq \sqrt[n]{|a_n|} \leq \sqrt[n]{|a_N|(\beta+\epsilon)^{-N}} \cdot (\beta+\epsilon).$$

Since $\lim_{n\to\infty} \sqrt[n]{|a_N|(\beta-\epsilon)^{-N}} = 1$, there exists $N' \in \mathbb{N}$ such that for any $n \geq N$,

$$\sqrt[n]{|a_N|(\beta-\epsilon)^{-N}} \geq (\beta-2\epsilon)/(\beta-\epsilon), \quad \sqrt[n]{|a_N|(\beta+\epsilon)^{-N}} \leq (\beta+2\epsilon)/(\beta+\epsilon).$$

Therefore, consider $N'' = \max\{N, N'\}$. Then for any $n \geq N''$,

$$\beta - 2\epsilon \leq \sqrt[n]{|a_n|} \leq \beta + 2\epsilon.$$

This completes the proof. □

The ratio and root test are especially useful for power series.

Definition 7.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence and $x \in \mathbb{R}$ be a real number. Then the series*

$$\sum_{n=0}^{\infty} a_n x^n$$

is called a power series with coefficients $\{a_n\}_{n\in\mathbb{N}}$.

Theorem 7.4. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Suppose*

$$\alpha = \lim_{n\to\infty} \sqrt[n]{|a_n|}, \quad R = 1/\alpha.$$

Then $\sum_{n=0}^{\infty} a_n x^n$ converges for any $|x| < R$ and diverges for any $|x| > R$.

8. Lecture 8: Absolute Convergence and Rearrangement

We have seen results regarding series of non-negative terms. For series with negative terms, the theory becomes more delicate. We now recall the notion of absolute convergence and conditional convergence.

Definition 8.1. *Let $\sum_{n=0}^{\infty} a_n$ be a series. We say that it absolutely converges if $\sum_{n=0}^{\infty} |a_n|$ converges, and it conditionally converges if it converges but does not absolutely converge.*

Proposition 8.1. *If $\sum_{n=0}^{\infty} a_n$ absolutely converges, then it converges.*

For a series that does not absolutely converge, one important convergence criterion is the following, due to Leibniz:

Theorem 8.2 (Leibniz)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Suppose*

(a) *for any $n \in \mathbb{N}$, $|a_n| \geq |a_{n+1}|$;*
(b) *for any $n \in \mathbb{N}$, $a_{2n-1} \leq 0, a_{2n} \geq 0$;*
(c) $\lim_{n\to\infty} a_n = 0$.

Then $\sum_{n=1}^{\infty} a_n$ converges.

Here is a question that we may ask ourselves: why shall we care about the distinction between absolute convergence and conditional convergence? The following theorem gives a reason: for an absolutely convergent sequence, the summation does not depend on the order of the terms, while surprisingly, for a conditionally convergent sequence, the summation does depend on the order of the terms!

Definition 8.2. *Let $\{n_k\}_{k\in\mathbb{N}}$ be a sequence of natural numbers so that every natural number appears once and only once. Then the series*

$$\sum_{k=1}^{\infty} a_{n_k}$$

is called a rearrangement of $\sum_{n=1}^{\infty} a_n$.

Theorem 8.3 (Riemann, 1868)**.** *Let $\sum_{n=1}^{\infty} a_n = s$ be absolutely convergent. Then for any rearrangement,*

$$\sum_{k=1}^{\infty} a_{n_k} = \sum_{n=1}^{\infty} a_n = s.$$

Theorem 8.4 (Riemann rearrangement theorem, 1868)**.** *Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. Then for any $s \in \mathbb{R} \cup \{\pm\infty\}$, there exists a rearrangement such that*

$$\sum_{k=1}^{\infty} a_{n_k} = s.$$

9. Lecture 9: Limits of Functions

Let $I \subset \mathbb{R}$ be an interval. Given a function $f : I \to \mathbb{R}$, we can discuss the limit of a function. Informally, $\lim_{x\to x_0} f(x) = y_0$ means that as x tends to x_0, $f(x)$ becomes infinitely close to y_0, or the difference of $f(x)$ and y_0 becomes infinitely small. B. Bolzano in 1817 first tried to make this idea rigorous:

Definition 9.1 (Bolzano, 1817, informal)**.** *Let $f : I \to \mathbb{R}$ be a function. Then we say that $f(x)$ converges to y_0 as x converges to x_0 if the difference $f(x)$ and y_0 can be made smaller than any quantity provided the difference between x_0 and x can be taken as small as we want.*

A. L. Cauchy in his 1820 book was using infinitely small quantities in his definitions. However, he interpreted limits via inequalities involving ϵ: that for an infinitesimal number $\epsilon > 0$, we can let δ be an infinitesimal number such that when x is in greater than $x_0 - \delta$ and less than $x_0 + \delta$, $f(x)$ will be greater than $y_0 - \epsilon$ and less than $y_0 + \epsilon$. Here, ϵ stands for the error. As we have already seen, Cauchy did not carefully use logic quantifiers. It was Weierstrass who in his 1865 lecture finally introduced the following definition of limits, which we use nowadays:

Definition 9.2 (Weierstrass, 1865)**.** *Let $f : I \to \mathbb{R}$ be a function. We say that $f(x)$ converges to y_0 as x goes to x_0 if for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $0 < |x - x_0| < \delta$, we have $|f(x) - y_0| \leq \epsilon$.*

Remark 9.1. *Based on the above definition, we say that $f(x)$ does not converge to y_0 if there exists $\epsilon > 0$, such that for any $\delta > 0$, there exists $0 < |x - x_0| < \delta$, such that we have $|f(x) - y_0| > \epsilon$.*

Similar to limits of sequences, we can prove various basic properties for limits of functions:

Theorem 9.1. *(a) If $f(x)$ converges to y_0 and y_0' as $x \to x_0$, then $y_0 = y_0'$. (b) If $f(x)$ converges as $x \to x_0$, then there exists a neighborhood $(x_0 - \delta, x_0 + \delta)$ such that its image $f(x_0 - \delta, x_0 + \delta)$ is bounded.*

Theorem 9.2. *Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions such that $\lim_{x\to x_0} f(x) = y_0$, $\lim_{x\to x_0} g(x) = z_0$.*

(a) $\lim_{x\to x_0}(f(x) \pm g(x)) = y_0 \pm z_0$.

(b) $\lim_{x\to x_0} f(x)g(x) = y_0 z_0$.

(c) $\lim_{x\to x_0} 1/f(x) = 1/y_0$ if $f(x) \neq 0$ and $y_0 \neq 0$.

The proof is similar to the case of sequences. However, we can also prove these properties using the relation between limits of functions and limits of sequences:

Theorem 9.3. *Let $f : I \to \mathbb{R}$ be a function. Then $f(x) \to y_0$ as $x \to x_0$ if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \neq x_0$ and $x_n \to x_0$, we have $f(x_n) \to y_0$.*

Proof. We only prove one direction. Suppose for any sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to x_0$, we have $f(x_n) \to y_0$. Then we show that $f(x) \to y_0$ as $x \to x_0$.

We prove by contrapositive. Suppose $f(x)$ does not converge to y_0 as $x \to x_0$. Then there exists $\epsilon > 0$, such that for any $\delta > 0$, there exists $0 < |x - x_0| < \delta$, such that

$$|f(x) - y_0| \geq \epsilon.$$

Let $\delta = 1/n$. Then we can find a sequence $0 < |x_n - x_0| < 1/n$ such that for any $n \in \mathbb{N}$,

$$|f(x_n) - y_0| \geq \epsilon.$$

However, $0 < |x_n - x_0| < 1/n$ implies that $x_n \to x$. This leads to a contradiction. $\square$

10. Lecture 10: Continuous Functions

Historically, mathematicians like Bolzano and Cauchy directly studied continuous functions using the $\epsilon - \delta$ language. In fact, the above definition of limits were only rephrased based on their description of continuous functions. Neither of them separately defined what a limit is.

Definition 10.1. *Let $I \subset \mathbb{R}$ be a subset. Then a function $f : I \to \mathbb{R}$ is continuous at $x_0 \in I$ if for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $|x - x_0| \leq \delta$, we have*

$$|f(x) - f(x_0)| \leq \epsilon.$$

Following the approach of Weierstrass, using limits of functions, we can now define continuous functions directly using limits (historically, it was Weierstrass who introduced the notation $\lim_{x \to x_0} f(x)$).

Definition 10.2. *Let $I \subset \mathbb{R}$ be a subset. Then a function $f : I \to \mathbb{R}$ is continuous at $x_0 \in I$ if $f(x) \to f(x_0)$ as $x \to x_0$.*

Here are some basic properties of continuous functions that can be deduced directly via properties of limits:

Theorem 10.1. *Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are continuous at x_0. Then $f + g, fg$ are also continuous at x_0. If $f(x) \neq 0$, then $1/f(x)$ is also continuous at x_0.*

Theorem 10.2. *Let $f : I \to J$ and $g : J \to \mathbb{R}$ be functions such that f is continuous at x_0 and g is continuous at $f(x_0)$. Then $g \circ f : I \to \mathbb{R}, x \mapsto g(f(x))$ is continuous at x_0.*

Proof. For any $\epsilon > 0$, since g is continuous at $f(x_0)$, there exists $\eta > 0$, such that for any $|y - f(x_0)| \leq \eta$, we have

$$|g(y) - g(f(x_0))| \leq \epsilon.$$

For the given $\eta > 0$, since f is continuous at x_0,there exists $\delta > 0$, such that for any $|x - x_0| \leq \delta$, we have

$$|f(x) - f(x_0)| \leq \eta,$$

which, by setting $y = f(x)$ in the first inequality, implies that $|g(f(x)) - g(f(x_0))| \leq \epsilon$. ☐

11. Lecture 11: Examples and Properties

We give some examples of continuous functions. In the following examples, the last two are more complicated as proving them involves properties on rational and irrational numbers. We will go back to them after discussing the theory of real numbers.

Theorem 11.1. *(a) The function*

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is discontinuous at $x = 0$.

(b) The function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is discontinuous at $x = 0$.

(c) (Dirichlet function) The function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is discontinuous at any $x \in \mathbb{R}$.

(d) (Riemann function) The function

$$f(x) = \begin{cases} \dfrac{1}{q}, & x = \dfrac{p}{q},\ p, q \in \mathbb{Z},\ gcd(p, q) = 1. \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \ \ or\ \ x = 0 \end{cases}$$

is discontinuous at any $x \neq 0$ but continous at $x = 0$.

Proof. (a) Consider $x_n = 1/n \to 0$. Then $f(x_n) = f(1/n) = n$ is unbounded and thus diverges. This implies that $f(x)$ diverges as $x \to 0$.

(b) Consider $x_n = 1/n\pi \to 0$. Then $f(x_n) = f(1/n\pi) = \sin(n\pi) = 0$. On the other hand, consider $x'_n = 1/(n\pi + \pi/2)$. Then Then $f(x'_n) = f(1/(n\pi + \pi/2)) = \sin(n\pi + \pi/2) = 1$. Since $f(x_n)$ and $f(x'_n)$ converge to different numbers, $f(x)$ diverges as $x \to 0$. □

For Examples (c) and (d), we need to know more about properties of real numbers in order to prove them:

Theorem 11.2. *For any $a, b \in \mathbb{R}$ with $a < b$, there exists $x \in \mathbb{Q}$ such that $x \in (a, b)$; there also exists $x \notin \mathbb{Q}$ such that $x \in (a, b)$.*

Using the above theorem, we can prove the function in (c) is discontinuous:

Proof. (c) We assume $x_0 \in \mathbb{Q}$ and leave the other case that $x_0 \notin \mathbb{Q}$ to the readers. Let $\epsilon < 1$. Then for any $\delta > 0$, we know there exists $x \in (x_0 - \delta, x_0 + \delta)$ such that $x \notin \mathbb{Q}$. Then

$$|f(x) - f(x_0)| = 1 > \epsilon.$$

Hence $f(x)$ is discontinuous at $x_0 \in \mathbb{Q}$. □

Finally, we list some properties of continuous functions. All these properties involves the theory of real numbers and thus all of the proofs will be postponed.

Theorem 11.3. *Let I be a (bounded) closed interval and $f : I \to \mathbb{R}$ be a continuous function. Then (a) the image $f(I)$ is bounded, and (b) there exists $a, b \in I$ such that for any $x \in I$,*

$$f(a) \leq f(x) \leq f(b).$$

We call $f(a)$ and $f(b)$ the minimum and maximum of f on the interval I.

Remark 11.1. *On an open interval, the properties may fail. For example, one can consider $I = (0, 1)$ and $f(x) = 1/x$. On a closed but unbounded interval, the properties may also fail. For example, one can consider $I = [0, +\infty)$ and $f(x) = x$.*

Theorem 11.4 (Intermediate value theorem, Bolzano, 1817 (Weierstrass, 1865))**.** *Let $f : I \to \mathbb{R}$ be a continuous function. For $a < b$, suppose $f(a) \geq c$ and $f(b) \geq c$. Then there exists $a \leq x \leq b$ such that $f(x) = c$.*

Finally, we explain a relevant but different property of continuous functions. Here is a question:

Let $f : I \to \mathbb{R}$ be a continuous function. Is it true that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $|x - x'| \leq \delta$, we have

$$|f(x) - f(x_0)| \leq \epsilon?$$

Unfortunately, the answer is no, because in the above definition, we require that $\delta > 0$ only depends on ϵ, but in the definition of continuity, $\delta > 0$ may depend on the point x_0 that we fix.

Nevertheless, this property is true for bounded closed intervals. Later, we will see that this is relavant in proving the boundedness of continuous functions:

Theorem 11.5. *Let I be a bounded closed interval and $f : I \to \mathbb{R}$ be a continuous function. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $|x - x'| \leq \delta$, we have*

$$|f(x) - f(x_0)| \leq \epsilon.$$

The proofs of all these theorems will be postponed after we discuss real numbers in the next two weeks.

12. Lecture 12: Rational Numbers and Order

We have seen a number of theorems that depend on delicate properties of real numbers that we did not discuss. Starting from this lecture, we will try to prove these theorems.

We will now define real numbers, study their properties, and eventually, prove all the theorems listed above. First, we start by recalling the set of rational numbers.

Definition 12.1 (Relation)**.** *Let A be a set. Suppose for any $a \in A$, there is an assignment of a subset of elements A. Then we say this assignment is a relation $\sim$ on A, and write $a \sim b$ if b is contained in the subset assigned to a.*

More formally, a relation is simply any subset of $A \times A = \{(a, b) \mid a \in A, b \in A\}$, where pairs inside this subset are written as $a \sim b$.

The most common relations we have seen in math are orders and equivalences:

Definition 12.2 (Order)**.** *Let A be a set. Then a relation $<$ on A is called an order if*

(1) *for any $a, b \in A$, then exactly one of the following statements holds:*

$$a < b, \quad a = b, \quad b < a.$$

(2) *for any $a, b, c \in A$, if $a < b$ and $b < c$, then $a < c$.*

Definition 12.3 (Equivalence relation)**.** *Let A be a set. Then a relation $\sim$ on A is called an equivalent relation if*

(1) *for any $a \in A$, we have $a \sim a$;*
(2) *for any $a \sim b$, we have $b \sim a$;*
(3) *for any $a \sim b$ and $b \sim c$, we have $a \sim c$.*

The set of all equivalence classes is the set where equivalent elements in A determine the same element.

More formally, the set of equivalence classes can be constructed from the set A as follows: For $a \in A$, we define an equivalence class to be

$$[a] = \{a' \in A \mid a \sim a'\}.$$

In particular, if $a \sim a'$, we know $[a] = [a']$. Then the set of equivalence classes to be

$$A/\sim = \{[a] \mid a \in A\}.$$

In other words, $A/\sim$ is a set whose elements are themselves subsets of A. This may look like a weird set, but it does what we want (by making $[a] = [a']$ an equality).

Rational numbers are exactly an example of sets defined by some equivalence classes (as we know that a rational number is represented by a pair of integers p/q in a non-unique way).

Definition 12.4 (Rational numbers)**.** *Let $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ be a pair of integers where $q \neq 0$. We say that $(p, q) \sim (p', q')$ if $pq' = qp'$. We define the equivalence class by p/q. Then the set of rational numbers $\mathbb{Q}$ is the set of all equivalence classes p/q.*

To summarize the properties of rational numbers, we can list a number of axioms which the rational numbers satisfy. Namely, they should include additions, subtractions, multiplications, divisions, and orders.

Definition 12.5. *A field is a set F together with two operations, addition and multiplication, satisfying the following axioms:*

(1) *Axioms for additions:*
   (a) *for $x, y \in F$, $x + y \in F$;*
   (b) *for $x, y \in F$, $x + y = y + x$;*
   (c) *for $x, y, z \in F$, $(x + y) + z = x + (y + z)$;*
   (d) *there exists $0 \in F$ such that for $x \in F$, $0 + x = x$;*
   (e) *for any $x \in F$, there exists $-x \in F$ such that $x + (-x) = 0$.*
(2) *Axioms for multiplications:*
   (a) *for $x, y \in F$, $xy \in F$;*
   (b) *for $x, y \in F$, $xy = yx$;*
   (c) *for $x, y, z \in F$, $(xy)z = x(yz)$;*
   (d) *there exists $1 \in F$, $1 \neq 0$, such that for $x \in F$, $1x = x$;*
   (e) *for any $x \in F$, $x \neq 0$, there exists $1/x \in F$ such that $x(1/x) = 1$.*
(3) *Axioms for distributions:*
   (a) *for $x, y, z \in F$, $(x + y)z = xz + yz$.*

Definition 12.6. *An ordered field F is a field with an order such that*

(1) *for any $x, y, z \in F$, if $x < y$, then $x + z < y + z$;*
(2) *for any $x, y, z \in F$, if $x < y$ and $z > 0$, then $xz < yz$.*

Theorem 12.1. *Let F be an ordered field. Then $\mathbb{N} \subset F$.*

Proof. For $n \in \mathbb{N}$, we define $n = 1 + 1 + \cdots + 1$ (n times) in the field F by mathematical induction. We need to show that for any $m \neq n \in \mathbb{N}$, we also have $m \neq n \in F$. Since F is an ordered field, we have the order $0 < 1$. When $m < n$ in $\mathbb{N}$, by mathematical induction we have $m < m+1 < \cdots < n$ in F. Then $m \neq n$ by the assumption that $<$ is an order. ☐

One property about rational numbers that we may have heard of is that rational numbers are dense. This is in fact closely related to the Archimedean property of rational numbers:

Definition 12.7 (Archimedean field)**.** *An ordered field F is called an Archimedean field if for any $x \in F$, there exists $n \in \mathbb{N}$ such that $n > x$.*

Theorem 12.2 (Density)**.** *Let F be an Archimedean field. Then for any $x, y \in F$ with $x < y$, there exists $z \in F$ such that $x < z < y$.*

Proof. We assume that $0 < x < y$. By the Archimedean property, we know that there exists $n \in \mathbb{N}$ such that $n > 1/(y - x)$. Since $y - x > 0$, this means that

$$n(y - x) > 1.$$

Then we show that there exists $m \in \mathbb{Z}$ such that

$$nx < m < ny.$$

Suppose this is not true. Then this means for any $m \in \mathbb{Z}$, either $m < nx$ or $m > ny$. We know there exists $m \in \mathbb{N}$ such that $m < nx$. Then

$$m + 1 < nx + 1 < ny.$$

However, we assumed that either $m+1 < nx$ or $m+1 > ny$. This then implies that for any $m \in \mathbb{N}$,

$$m + 1 < nx,$$

which contradicts the Archimedean property. $\square$

Theorem 12.3. *$\mathbb{Q}$ is an Archimedean ordered field.*

13. Lecture 13: Desired Properties of Real Numbers

It seems that $\mathbb{Q}$ already satisfies a number of nice properties. However, the following example shows that rational numbers do not allow all operations we may want; for instance, we cannot always take square roots:

Theorem 13.1. *There does not exist a rational number $x \in \mathbb{Q}$ such that $x^2 = 2$.*

Proof. We prove by contrapositive. Suppose there exists $p, q \in \mathbb{Z}$ with no common factors greater than 1, such that we have $(p/q)^2 = 2$. Then

$$p^2 = 2q^2.$$

This implies that p must be even. Thus, we can write $p = 2k$ for some $k \in \mathbb{Z}$. Now

$$4k^2 = 2q^2, \quad 2k^2 = q^2.$$

This implies that q must also be even, but then p, q has a common factor 2, which is a contradiction. $\square$

The above example suggests that we need to enlarge the field of rational numbers $\mathbb{Q}$ (in order to allow more operations, for example taking square roots). This means that real numbers $\mathbb{R}$ should satisfy even more properties than $\mathbb{Q}$.

The goal of constructing real numbers therefore consists of two parts: First, we need to propose an additional property that real number should further satisfy (to ensure more operations are allowed, for example taking square roots). Then we need to find a way to construct real numbers (to ensure that the additional property can be satisfied).

Historically, two approaches of constructing real numbers were introduced in 1872, by R. Dedekind and G. Cantor independently. First, we will introduce R. Dedekind's approach.

The idea is to impose the additional property on the order of the field. In order to do that, let us take a more careful look at the example of square roots:

Theorem 13.2. *For any $x \in \mathbb{Q}$ such that $x^2 < 2$, there exists $y \in \mathbb{Q}$ such that $x^2 < y^2 < 2$.*

Proof. Let $x \in \mathbb{Q}$ be such that $x^2 < 2$. We assume that $x > 0$. Since $x^2 < 2$, we know $x < 2$. By the Archimedean property of $\mathbb{Q}$, we know that there exists $m \in \mathbb{N}$, such that

$$x^2 < x^2 + \frac{1}{m} < 2.$$

Therefore, consider $n \in \mathbb{N}$ such that $n = 5m$. Then we have

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{1}{n}\left(2x + \frac{1}{n}\right) < x^2 + \frac{5}{n} = x^2 + \frac{1}{m} < 2.$$

$\square$

We can try to make the above property formal. This property of real numbers was first realized by Bolzano in 1817, who was working in relative isolation from most European mathematicians at that time (and was only rediscovered later after his death).

Definition 13.1 (Bolzano, 1817). *Let A be an ordered set and $B \subset A$ be a subset. Then an upper bound of B is an element $a \in A$ such that for any $b \in B$, $b \leq a$. The supremum (or least upper bound) of B is an element $c \in A$ such that c is an upper bound of B, and for any upper bound $a \in A$ of B, $a \geq c$. If it exists, we denote it by* $\sup B$. *Similarly, we denote the infimum (greatest lower bound) of B, if it exists, by* $\inf B$.

Theorem 13.3. *The set $\{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a least upper bound in $\mathbb{Q}$.*

14. Lecture 14: Real Numbers by Dedekind Cuts I

We will define real numbers $\mathbb{R}$ by filling in such gaps, requiring that any bounded subset has a least upper bound.

Theorem 14.1 (Dedekind, 1872). *There exists an Archimedean ordered field $\mathbb{R}$ that satisfies the least-upper-bound property: any bounded subset has a least upper bound.*

Proof. We will construct the set of real numbers $\mathbb{R}$ from rational numbers by adding in all the possible least upper bounds in $\mathbb{Q}$. In rational numbers, we have seen that the least upper bound does not always exist, so instead we need to represent the least upper bounds by subsets of rational numbers.

Define a Dedekind cut to be a subset $\alpha \subset \mathbb{Q}$ that satisfies the following property:

(1) $\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$;
(2) if $p \in \alpha$ and $q < p$, then $q \in \alpha$;
(3) if $p \in \alpha$, then there exists $r \in \alpha$ such that $p < r$.

We define $\mathbb{R}$ to be the set of all Dedekind cuts of $\mathbb{Q}$. (For any $\alpha \in \mathbb{Q}$, it defines a Dedekind cut by

$$\alpha = \{x \in \mathbb{Q} \mid x < \alpha\}.$$

One can verify this is a Dedekind cut, so in this sense $\mathbb{Q}$ is identified with a subset of $\mathbb{R}$.)

Step 0: We define the order on $\mathbb{R}$ and verify the axioms of an order. We define that

$$\alpha < \beta$$

if $\alpha \subset \beta$ and $\alpha \neq \beta$.

If $\alpha \not\subset \beta$, we need to show that $\beta \subset \alpha$. Since $\alpha \not\subset \beta$, there exists $x \in \beta$ with $x \notin \alpha$. For $y \in \alpha$, if $y > x$, then $x \in \alpha$ as well, which is a contradiction. So for any $y \in \alpha$, $y < x$ and hence $y \in \beta$. This implies that if $\alpha \not\subset \beta$, then

$$\beta \subset \alpha.$$

If $\alpha \subset \beta$ and $\beta \subset \gamma$, it is obvious that $\alpha \subset \gamma$. This shows that $<$ defines an order on $\mathbb{R}$.

Step 1: We verify the Archimedean property. For any $\alpha \in \mathbb{R}$, we show that there exists $n \in \mathbb{N}$ such that $\alpha \subset n$, in other words, for any $x \in \alpha$, $x < n$. In fact, by assumption, $\alpha \neq \mathbb{Q}$, so there exists $y \notin \alpha$. This means for any $x \in \alpha$, $x < y$ as otherwise,

$$y < x, \quad y \in \alpha.$$

By the Archimedean principle for $\mathbb{Q}$, there exists $n \in \mathbb{N}$ such that $n > y$. Then for any $x \in \alpha$, $n > y > x$.

Step 2: We verify the least-upper-bound property. For any non-empty subset $A \subset \mathbb{R}$, suppose A has an upper bound β. We show that A has a least upper bound γ. We consider

$$\gamma = \bigcup_{\alpha \in A} \alpha.$$

First, this is a cut because (1) $\gamma \subset \beta$ implies $\gamma \neq \mathbb{Q}$, and $\alpha \subset \gamma$ implies $\gamma \neq \varnothing$; (2) if $p \in \gamma$, this means there exists $\alpha \in A$ such that $p \in \alpha$, and hence for any $q < p$, $q \in \alpha$; (3) if

$p \in \gamma$, then there exists $\alpha \in A$ such that $p \in \alpha$, which implies that there exists $r \in \alpha \subset \gamma$ such that $p < r$.

Second, this is the least upper bound of A because: (1) γ is an upper bound of A, since for any $\alpha \in A$,

$$\alpha \subset \bigcup_{\alpha \in A} \alpha = \gamma.$$

(2) for any upper bound β of A, $\gamma \leq \beta$, since β is an upper bound means that for any $\alpha \in A$, $\alpha \subset \beta$, which implies that

$$\gamma = \bigcup_{\alpha \in A} \alpha \subset \beta.$$

This shows that the least-upper-bound property holds. □

15. Lecture 15: Real Numbers by Dedekind Cuts II

Theorem 15.1 (Dedekind, 1872)**.** *There exists an Archimedean ordered field* $\mathbb{R}$ *that satisfies the least-upper-bound property: any bounded subset has a least upper bound.*

Proof Continued. **Step 3:** We define the addition and prove the axioms. Define the addition to be

$$\alpha + \beta = \{x + y \mid x \in \alpha, y \in \beta\}.$$

First, we show that $\alpha + \beta$ is always a cut. This is because: (1) $\alpha + \beta \neq \varnothing$ since both of them are non-empty, and $\alpha + \beta \neq \mathbb{Q}$ since there exists $x \notin \alpha$, $y \notin \beta$, and this implies that

$$x + y \notin \alpha + \beta$$

(for any $x' \in \alpha$, we know $x < x'$, and for any $y' \in \beta$, we know $y < y'$, which implies that for any $x' \in \alpha$ and $y' \in \beta$, $x' + y' \neq x + y$). (2) For any $p \in \alpha + \beta$, there exists $x \in \alpha, y \in \beta$ such that $p = x + y$. If $q < p$, we can write

$$q = (x - p + q) + y.$$

Since $x - p + q < x$, we know $x - p + q \in \alpha$. (3) For any $p \in \alpha + \beta$, there exists $x \in \alpha, y \in \beta$ such that $p = x + y$. Since there exists $x' \in \alpha$ such that $x < x'$, we know

$$q = x' + y \in \alpha + \beta, \quad p < q.$$

Second, we show that $\alpha + \beta = \beta + \alpha$ and $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$. These are obvious.

Then, we define $0 = \{x \in \mathbb{Q} \mid x < 0\}$ and show that $0 + \alpha = \alpha$. For any $x \in \alpha$ and $y < 0$,

$$x + y < x, \quad x + y \in \alpha.$$

This implies that $0 + \alpha \subset \alpha$. On the other hand, for any $x \in \alpha$, there exists $z \in \alpha$ such that $x < z$. Then we can write

$$x = (x - z) + z \in 0 + \alpha.$$

This implies that $\alpha \subset 0 + \alpha$ and therefore $0 + \alpha = \alpha$.

Finally, for any $\alpha \in \mathbb{R}$, if $\alpha \notin \mathbb{Q}$, we define

$$-\alpha = \{x \in \mathbb{Q} \mid \text{for any } y \in \alpha,\ x + y < 0\},$$

and show that $\alpha + (-\alpha) = 0$. For any $x \in \alpha$ and $y \in -\alpha$,

$$x + y < 0, \quad x + y \in 0.$$

This implies that $\alpha + (-\alpha) \subset 0$. On the other hand, for any $-z < 0$, there exists $x \in \alpha$ such that $x + z \notin \alpha$, which means $y \in \alpha$, $y < x + z$. (Otherwise, by mathematical induction, we know that for any $n \in \mathbb{N}$, $x + nz \in \alpha$, but then by the Archimedean property of $\mathbb{Q}$ this means that $\alpha = \mathbb{Q}$, which is a contradiction.) Then we have

$$-z = x + (-x - z).$$

We claim that $-x - z \in -\alpha$. For any $y \in \alpha$, since $y < x + z$,

$$(-x - z) + y < 0.$$

This implies that $0 \subset \alpha + (-\alpha)$ and therefore $0 = \alpha + (-\alpha)$.

Step 4: We define multiplications and prove the axioms. This is harder to define. When $\alpha, \beta \geq 0$, we define

$$\alpha \cdot \beta = \{xy \mid x \in \alpha, y \in \beta, x, y > 0\} \cup \{r \mid r < 0\}.$$

When $\alpha < 0$ and $\beta \geq 0$, we define

$$\alpha \cdot \beta = (-\alpha) \cdot \beta.$$

When $\alpha, \beta < 0$, we define

$$\alpha \cdot \beta = (-\alpha) \cdot (-\beta).$$

For simplicity, however, we will only verify the properties of multiplications assuming $\alpha, \beta, \gamma \geq 0$, and leave the rest of the cases to the readers.

First, we need to show that

$$\alpha \cdot \beta = \{xy \mid x \in \alpha, y \in \beta, x, y > 0\} \cup \{z \mid z < 0\}$$

is indeed a Dedekind cut. (1) We know $\alpha\beta \neq \varnothing$ because $\alpha, \beta \neq \varnothing$. Since there exists $x \notin \alpha$, $y \notin \beta$, $x > x'$ for any $x' \in \alpha$ and $y > y'$ for any $y' \in \alpha$ (as explained above). Moreover, since $0 \in \alpha, \beta$, we know $x, y \geq 0$. Therefore, for any $x' \in \alpha, y' \in \beta$ with $x', y' \geq 0$,

$$xy > x'y', \quad xy \notin \alpha\beta.$$

(2) For any $r \in \alpha\beta$ and $s < r$, we need to show $s \in \alpha\beta$. Suppose $s \leq 0$. Then this is obvious. Now suppose $0 < s < r$. Then $r \in \alpha\beta$ means there exists $x \in \alpha, y \in \beta$, $x, y > 0$ such that $r = xy$. Then

$$s = x(s/x), \quad x \in \alpha, \quad s/x < r/x = y \in \beta.$$

This implies $s \in \alpha\beta$. (3) For any $r \in \alpha\beta$, we need to show that there exists $s > r$ such that $s \in \alpha\beta$. Again, suppose $r \leq 0$. Then this is obvious. Now suppose $0 < r$. There exists $x \in \alpha, y \in \beta$, $x, y > 0$ such that $r = xy$. Note that for $y \in \beta$, there exists $y' > y$ such that $y' \in \beta$. Then let

$$s = xy' > xy = r, \quad x \in \alpha, y' \in \beta.$$

This implies that $s > r$ and $s \in \alpha\beta$. Now, the above three properties show that $\alpha\beta$ is indeed a Dedekind cut.

Second, we show that $\alpha\beta = \beta\alpha$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$. These are obvious.

Next, we show that $1\alpha = \alpha$. Since $0 \subset \alpha$, we know $0 \subset 1\alpha$ by definition of multiplication. Now, pick $r \in 1\alpha$ and $r \geq 0$. We know there exists $0 \leq x < 1$ and $y \in \alpha$ with $y \geq 0$ such that $r = xy$. So

$$r = xy < y \in \alpha, \quad r \in \alpha.$$

This implies $1\alpha \subset \alpha$. On the other hand, pick $r \in \alpha$ with $r \geq 0$. There exists $s > r$ with $s \in \alpha$. Then $0 < r/s < 1$. We thus have

$$(r/s) < 1, \quad s \in \alpha, \quad r = (r/s)s \in 1\alpha.$$

This implies that $\alpha \subset 1\alpha$ and hence $1\alpha = \alpha$.

Then, we show that for any $0 \subset \alpha$ with $0 \neq \alpha$, there exists $(1/\alpha)$ such that $\alpha(1/\alpha) = 1$. We define

$$1/\alpha = \{1/y \mid \text{for any } x \in \alpha,\ y > x\} \cup \{r \mid r < 0\}.$$

We leave the readers to verify that this is a Dedekind cut. Then for any $x \in \alpha, 1/y \in 1/\alpha$, since $y > x$,

$$x(1/y) = x/y < 1.$$

This implies $\alpha(1/\alpha) \subset 1$. On the other hand, for any $r < 1$, we will find $x \in \alpha$ and $1/y \in 1/\alpha$ such that $x/y = r$. This means that we need to show there exists

$$x \in \alpha, \quad \text{for any } y \in \alpha, \ x/r > y.$$

Suppose this is not the case. Then for any $x \in \alpha$, $x/r \in \alpha$. By mathematical induction, $x/r^n \in \alpha$ for any $n \in \mathbb{N}$. Using the binomial expansion, we know that

$$x\frac{1}{r^n} = x\left(1 + \frac{1-r}{r}\right)^n \geq nx\frac{1-r}{r}.$$

By the Archimedean principle, for any $y \in \mathbb{Q}$, there exists $n \in \mathbb{N}$ such that $x/r^n > y$, which implies that $y \in \alpha$ and leads to a contradiction $\alpha = \mathbb{Q}$. Therefore, we conclude that there exists

$$x \in \alpha, \quad \text{for any } y \in \alpha, \ x/r > y.$$

Thus, $r = x(x/r) \in \alpha(1/\alpha)$. This implies that $1 \subset \alpha(1/\alpha)$. Hence $\alpha(1/\alpha) = 1$.

Step 5: We prove the axioms of an ordered field.

First, suppose $\alpha \subset \beta$. It is obvious that $\alpha + \gamma \subset \beta + \gamma$. We leave it to the readers. Second, suppose $\alpha \subset \beta$ and $0 \subset \gamma$. We will show that $\alpha\gamma \subset \beta\gamma$. Suppose $0 \subset \alpha \subset \beta$. Then

$$\alpha\gamma = \{xz \mid x \in \alpha, z \in \gamma, x, y > 0\} \cup \{r \mid r < 0\}.$$

Since $\alpha \subset \beta$, we know that if $x \in \alpha$ then $x \in \beta$, so

$$\alpha\gamma \subset \{xz \mid x \in \beta, z \in \gamma, x, y > 0\} \cup \{r \mid r < 0\} = \beta\gamma.$$

For the other cases, we leave it to the readers again. □

16. Lecture 16: Real Numbers by Cauchy Sequences I

We now introduce G. Cantor's approach to the definition of real numbers. G. Cantor considered to impose a different (but equivalent) property of the field, which does not rely on the order.

Again, let us take a more careful look at the example of square roots, rephrasing the same property, but in a slightly different way:

Theorem 16.1. *For any $n \in \mathbb{Q}$, there exists $x \in \mathbb{Q}$ such that $|x^2 - 2| \leq 5/n$.*

Proof. By the Archimedean property, we show that for any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that

$$\left(\frac{m}{n}\right)^2 < 2 < \left(\frac{m+1}{n}\right)^2.$$

In fact, we know $(0/n)^2 < 2$, and by the Archimedean property, there exists $m \in \mathbb{N}$ such that $(m/n)^2 \geq m/n^2 > 2$. Therefore, we can consider the maximal $m \in \mathbb{N}$ such that $(m/n)^2 < 2$. Moreover, we know $m < 2n$. Then $((m+1)/n)^2 > 2$. Therefore,

$$|x^2 - 2| = \left|\left(\frac{m}{n}\right)^2 - 2\right| \leq \left|\left(\frac{m}{n}\right)^2 - \left(\frac{m+1}{n}\right)^2\right| = \frac{2m+1}{n} \leq \frac{5}{n}.$$

□

Theorem 16.2. *On the set of rationals $\mathbb{Q}$, there exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ that does not converge in $\mathbb{Q}$ but $\{x_n^2\}$ converges to 2.*

Proof. For any $n \in \mathbb{N}$, we let $x_n = m/n$ determined in the above proposition such that

$$|x_n^2 - 2| \leq \frac{5}{n}.$$

Then we know $\{x_n^2\}$ converges to 2. We show that $\{x_n\}$ is a Cauchy sequence. This is because for any $\epsilon > 0$, we can choose $N \geq 2/\epsilon$, and then for any $m, n \geq N$,

$$|x_n - x_m| \leq |x_n - 2| + |x_m - 2| \leq \frac{1}{n} + \frac{1}{m} \leq \epsilon.$$

It does not converge to a rational number because if it converges then the limit x satisfies $x^2 = 2$, but no rational number squares to 2. □

17. Lecture 17: Real Numbers by Cauchy Sequences II

We will define real numbers $\mathbb{R}$ by filling in such gaps, by now directly requiring that all the Cauchy sequences converges:

Theorem 17.1 (Cantor, 1876)**.** *There exists an Archimedean ordered field $\mathbb{R}$ that satisfies the completeness property: any Cauchy sequence converges.*

Proof. We will construct the real numbers $\mathbb{R}$ from the rational numbers by adding in all the possible limits of Cauchy sequences. Limits of Cauchy sequences in $\mathbb{Q}$ do not always exist, so we will represent them by the Cauchy sequences themselves.

Consider the set of all Cauchy sequences in $\mathbb{Q}$. Define an equivalence relation as follows: for two Cauchy sequences, we say $\{a_n\}_{n\in\mathbb{N}} \sim \{b_n\}_{n\in\mathbb{N}}$ if

$$\lim_{n\to\infty}(a_n - b_n) = 0.$$

Then we define $\mathbb{R}$ to the set of all the equivalence classes of Cauchy sequences in $\mathbb{Q}$.

Step 0: We define additions and multiplications. Define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}, \quad \{a_n b_n\} = \{a_n b_n\}.$$

In order to say that the addition and multiplications are defined on the equivalence classes, we need to say that for different Cauchy sequences that are equivalent, the sums or products we get are also equivalent. For example, if $\{a_n\} \sim \{a'_n\}$, i.e. $a_n - a'_n \to 0$, we know that $\{a_n + b_n\} \sim \{a'_n + b_n\}$ as

$$(a_n + b_n) - (a'_n + b_n) = a_n - a'_n \to 0, \quad n \to \infty.$$

Moreover, as any Cauchy sequence $\{b_n\}$ must be bounded in $\mathbb{Q}$ (this will be left as an exercise), we also know that $\{a_n b_n\} \sim \{a'_n b_n\}$ as

$$a_n b_n - a'_n b_n = (a_n - a'_n) b_n \to 0, \quad n \to \infty.$$

We also leave it to the readers to show that the sum of Cauchy sequences are still Cauchy, and the products of Cauchy sequences are still Cauchy.

Then all the axioms of additions and multiplications are obvious.

Step 1: We define the order as follows. We define

$$\{a_n\} < \{b_n\}$$

if $\{a_n\} \not\sim \{b_n\}$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $a_n < b_n$. Again, we need to show that this does not depend on the representative of the Cauchy sequence we choose.

First, we show that $\{a_n\} < \{b_n\}$ implies the following stronger condition: that there exists $\epsilon_0 \in \mathbb{Q}$, $\epsilon_0 > 0$ and $N \in \mathbb{N}$ such that for any $n \geq N$,

$$a_n \leq b_n - \epsilon_0.$$

Otherwise, for any $\epsilon > 0$ and $N \in \mathbb{N}$, there exists $n \geq N$ such that

$$b_n - \epsilon/3 < a_n < b_n, \quad |a_n - b_n| < \epsilon/3.$$

However, since $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, there exists $N' \in \mathbb{N}$ such that for any $m, n \geq N$,

$$|a_n - a_m| \leq \epsilon/3, \quad |b_n - b_m| \leq \epsilon/3.$$

This will imply that for any $N = N'$, there exists $n \geq N$ such that for any $m \geq N$, we have

$$|a_m - b_m| \leq |a_m - a_n| + |a_n - b_n| + |b_n - b_m| < \epsilon.$$

Which shows that $\{a_n\} \sim \{b_n\}$ and leads to a contradiction. From now on, we will freely use this stronger condition for $\{a_n\} < \{b_n\}$.

Given the above condition, we will now show that the order does not depend on the representative of the Cauchy sequence we choose: If $\{a_n\} \sim \{a'_n\}$ and $\{a_n\} < \{b_n\}$, then we also have $\{a'_n\} < \{b_n\}$. Suppose there exists $\epsilon_0 \in \mathbb{Q}$, $\epsilon_0 > 0$ and $N \in \mathbb{N}$ such that for any $n \geq N$,

$$a_n \leq b_n - \epsilon_0.$$

Since $\{a_n\} \sim \{a'_n\}$, there exists $N' \in \mathbb{N}$ such that for any $n \geq N'$,

$$a_n - \epsilon_0/2 < a'_n \leq a_n + \epsilon_0/2.$$

Combining the two inequalities, we can show that for any $n \geq \max\{N, N'\}$,

$$a'_n \leq a_n + \epsilon_0/2 \leq b_n - \epsilon_0/2.$$

Hence $\{a'_n\} < \{b_n\}$, so the order does not depend on which Cauchy sequence we choose.

To show that $<$ defines an order. We need to prove that if $\{a_n\} \not< \{b_n\}$ and $\{a_n\} \not\sim \{b_n\}$, then we must have $\{b_n\} < \{a_n\}$. $\{a_n\} \not< \{b_n\}$ means for any $N \in \mathbb{N}$, there exists $n_1 \geq N$,

$$a_{n_1} \geq b_{n_1}.$$

$\{a_n\} \not\sim \{b_n\}$ means there exists $\epsilon > 0$, such that for any $N \in \mathbb{N}$, there exists $n_2 \geq N$, such that

$$|a_{n_2} - b_{n_2}| \geq \epsilon.$$

Since $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, for any $\epsilon > 0$, there exists $N' \in \mathbb{N}$ such that for any $m, n \geq N'$, we have

$$|a_n - a_m| \leq \epsilon/5, \quad |b_n - b_m| \leq \epsilon/5.$$

Now, we choose $N = N'$ and choose any $n \geq N$. Then since $n_1, n_2 \geq N$, we can combine the inequalities and get

$$a_{n_1} - b_{n_1} = |a_{n_1} - b_{n_1}| \geq |a_{n_2} - b_{n_2}| - |a_{n_1} - a_{n_2}| - |b_{n_2} - b_{n_1}| \geq 3\epsilon/5.$$

Hence for any $n \geq N$, we have

$$a_n - b_n \geq (a_n - a_{n_1}) + (a_{n_1} - b_{n_1}) + (b_{n_1} - b_n) \geq \epsilon/5.$$

Therefore, we can conclude that $\{b_n\} < \{a_n\}$.

The rest of the axioms of orders are obvious.

Step 2: We prove the Archimedean property, that for any Cauchy sequence $\{a_n\}$, there exists $m \in \mathbb{N}$ such that $\{m\} \geq \{a_n\}$. Since any Cauchy sequence is bounded (left as an exercise), there exists $r \in \mathbb{Q}$ such that for any $n \in \mathbb{N}$, $r \geq a_n$. By the Archimedean principle for $\mathbb{Q}$, there exists $m \in \mathbb{N}$ such that $m \geq r \geq a_n$. By the definition of the order, this implies that $\{m\} \geq \{a_n\}$ which finishes the proof.

Step 3: We prove that any Cauchy sequence in $\mathbb{R}$ converges: Consider a sequence in $\mathbb{R}$, which is a sequence of Cauchy sequences

$$\{a_{n1}\}, \{a_{n2}\}, \{a_{n3}\}, \ldots, \{a_{nk}\}, \{a_{n,k+1}\}, \ldots.$$

Suppose for any $\epsilon > 0$, there exists $K \in \mathbb{N}$, such that for any $k, l \geq K$, we have

$$|\{a_{nk}\} - \{a_{nl}\}| \leq \epsilon.$$

Then we need to conclude that $\lim_{k\to\infty}\{a_{nk}\} = \{c_n\}$ for some Cauchy sequence $\{c_n\}$. This means for any $\epsilon > 0$, there exists $K \in \mathbb{N}$, such that for any $k \geq K$, we have

$$|\{a_{nk}\} - \{c_n\}| \leq \epsilon.$$

We define $\{c_n\}$ using mathematical induction as follows. Let $\epsilon = 1/m$ and consider the corresponding $K_m \in \mathbb{N}$ as above. Then for any $k \geq K_m$, there exists $N_{m,k} \in \mathbb{N}$, such that for any $n \geq N_{m,k}$,

$$|a_{nk} - a_{nK_m}| \leq 1/m.$$

In particular, this is true for any $K_m \leq k < K_{m+1}$. Let $N'_m = \max\{N_{m,k} \mid K_m \leq k < K_{m+1}\}$. Therefore, we define

$$c_n = a_{nK_m}, \quad N'_m \leq n < N'_{m+1}.$$

First, we show that $\{c_n\}$ is a Cauchy sequence (this is non-trivial). Under this definition of $\{c_n\}$, we know that for $\epsilon = 1/m$, we can find $K_m \in \mathbb{N}$ such that for any $k \geq K_m$,

$$|\{a_{nk}\} - \{c_n\}| \leq \sup_{l\geq m}\big(\sup_{N'_l \leq n < N'_{l+1}} |a_{nk} - a_{nK_l}|\big) \leq 1/m.$$

This implies that $\{a_{nk}\}$ converges to $\{c_n\}$ as $k \to \infty$. (Question: why is it enough to choose $\epsilon = 1/m$ to show the convergence?) □

Finally, we discuss the relationship between Cauchy sequences and infinite decimal numbers. For any infinite decimal number $a_0.a_1a_2a_3\ldots a_n\ldots$ (of base 10), we can write it as an infinite series

$$a_0.a_1a_2a_3\ldots a_n\cdots = \sum_{n=0}^{\infty} \frac{a_n}{10^n},$$

where $a_n \in \mathbb{N}$ and $0 \leq a_n \leq 9$ for any $n \geq 1$. We can show that the partial sum is always a Cauchy sequence. In fact, we can also show that any Cauchy sequence is equivalent to an infinite decimal number.

This provides yet another model for the real numbers. The reason we did not choose this model is because this depends on base 10 numbers, but one can of course choose base 2, 3 or any other number. Therefore, the notion of Cauchy sequence is a generalization of infinite decimals that no longer depend on the base.

18. Lecture 18: Cauchy Sequences Revisited

We have seen two constructions of the real numbers, satisfying either the completeness property or the least-upper-bound property. We will prove that either equivalence implies the other:

Theorem 18.1. *A complete Archimedean ordered field where any Cauchy sequence converges satisfies the least-upper-bound property.*

Proof. Consider a non-empty subset $A \subset F$ with an upper bound. Then since F is an Archimedean ordered field, for any $n \in \mathbb{N}$, there exists a non-increasing $x_n \in F$ such that x_n is an upper bound of A but $x_n - 1/n$ is not an upper bound of A. Let x_1 be any upper bound. We can define x_n inductively as follows. Consider an upper bound x_{n-1}. Then by the Archimedean property, there exists $m \in \mathbb{N}$ such that $x_{n-1} - m/n$ is an upper bound of A but $x_{n-1} - (m+1)/n$ is not. Then we define $x_n = x_{n-1} - m/n$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Since F is complete, we may assume that $x_n \to x$. We now show that $x = \sup A$. First, for any $a \in A$, $x_n \geq a$, so $x \geq a$ as well. Second, since x_n is an upper bound of A but $x_n - 1/n$ is not an upper bound of A, there exists $a_n \in A$ such that

$$x_n - \frac{1}{n} \leq a_n \leq x_n.$$

Thus $a_n \to a$, and there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$x - \epsilon \leq a_n \leq x.$$

Hence $x = \sup A$. □

Theorem 18.2. *For an Archimedean ordered field that satisfies the least-upper-bound property, a bounded monotonely increasing (or decreasing) sequence converges.*

Proof. Consider a bounded monotonely increasing sequence $\{a_n\}_{n\in\mathbb{N}}$. Let $a = \sup\{a_n \mid n \in \mathbb{N}\}$. We claim that $a_n \to a$. In fact, since $a = \sup\{a_n \mid n \in \mathbb{N}\}$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$a_N \geq a - \epsilon.$$

Since $\{a_n\}_{n\in\mathbb{N}}$ is increasing, we know for any $n \geq N$,

$$a - \epsilon \leq a_n \leq a.$$

This shows that $a_n \to a$. □

Theorem 18.3. *An Archimedean ordered field that satisfies the least-upper-bound property: any Cauchy sequence converges.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence. We know that it is bounded. Define

$$b_n = \sup\{a_k \mid k \geq n\}, \quad c_n = \inf\{a_k \mid k \geq n\}.$$

Then $\{b_n\}_{n\in\mathbb{N}}$ is decreasing. This is because $\sup\{a_k \mid k \geq n+1\} \subset \sup\{a_k \mid k \geq n\}$, so

$$b_{n+1} = \sup\{a_k \mid k \geq n+1\} \leq \sup\{a_k \mid k \geq n\} = b_n.$$

Similarly, $\{c_n\}_{n\in\mathbb{N}}$ is increasing. Therefore, we may assume that $b_n \to b$ and $c_n \to c$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$, $|a_m - a_n| \leq \epsilon$. This means that for any $n \geq N$,

$$|a_n - b| \leq |a_n - b_n| + |b_n - b| \leq 2\epsilon, \quad |a_n - c| \leq |a_n - c_n| + |c_n - c| \leq 2\epsilon.$$

This means that $a_n \to b = c$. □

The notions introduced in the last theorem are useful for general sequences:

Definition 18.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Define*

$$b_n = \sup\{a_k \mid k \geq n\}, \quad c_n = \inf\{a_k \mid k \geq n\}.$$

We define $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ and $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} c_n$.

19. Lecture 19: Subsequences and Continuous Functions

When discussing sequence, we have seen that any convergence sequence is bounded, but not any bounded sequences converge. However, we can still prove the following statement.

Theorem 19.1 (Bolzano, 1817; Weierstrass, 1865)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Then there exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ that converges.*

Proof. The idea is to look at smaller and smaller intervals that contain infinitely many terms of $\{a_n\}$. Since $\{a_n\}$ is bounded, we can assume that there exists $b_0 < c_0$ such that for any $n \in \mathbb{N}$,

$$a_n \in [b_0, c_0].$$

We define a_{n_1} to be any term and thus $a_{n_1} \in [b_0, c_0]$. Let $d_0 = (b_0+c_0)/2$. Then either $[b_0, d_0]$ or $[d_0, c_0]$ contains infinitely many terms of $\{a_n\}$. Let $[b_1, c_1]$ be interval with infinitely many terms of $\{a_n\}$. Then by mathematical induction, we can choose intervals $[b_n, c_n]$ such that for any $n \in \mathbb{N}$, $[b_n, c_n] \subset [b_{n-1}, c_{n-1}]$, and $[b_n, c_n]$ contains infinitely many terms in $\{a_n\}$.

Let a_{n_1} be any term with

$$a_{n_1} \subset [b_0, c_0].$$

Since $[b_k, c_k]$ contains infinitely many terms in $\{a_n\}$, there exists $n_k > n_{k-1}, \dots, n_2, n_1$ such that

$$a_{n_{k+1}} \subset [b_k, c_k].$$

In particular, since $[b_k, c_k] \subset [b_{k-1}, c_{k-1}]$, we know for any $k \geq K + 1$,

$$a_{n_k} \subset [b_K, c_K].$$

Since $c_k - b_k = (c_0 - b_0)/2^k$, we know that $\{a_{n_k}\}_{k\in\mathbb{N}}$ is a Cauchy sequence as for any $K \in \mathbb{N}$, when $k, l \geq K + 1$,

$$|a_{n_k} - a_{n_l}| \leq c_K - b_K = \frac{(c_0 - b_0)}{2^K}.$$

and hence $\{a_{n_k}\}_{k\in\mathbb{N}}$ converges. □

The idea to look at smaller and smaller intervals and then find the potential limit can be formalized into the following idea:

Theorem 19.2. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of bounded intervals on $\mathbb{R}$ such that for any $n \in \mathbb{N}$, $I_{n+1} \subset I_n$. Then*

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

When the length of the intervals $\operatorname{diam}(I_n) \to 0$*, then $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one point.*

This idea is the key to proving the intermediate value theorem, which was first written down by Bolzano and Weierstrass.

Theorem 19.3 (Intermediate value theorem; Bolzano, 1817; Weierstrass, 1865)**.** *Let I be an interval and $f : I \to \mathbb{R}$ be a continuous function. Suppose $a, b \in I$, $a < b$ and*

$$f(a) \leq c, \quad f(b) \geq c.$$

Then there exists $x \in [a, b]$ such that $f(x) = c$.

Proof. Consider the interval $[a_1, b_1] = [a, b]$. Let $d_1 = (a_1 + b_1)/2$. Then either $f(d_1) \geq c$ or $f(d_1) \leq c$. When $f(d_1) \geq c$, we set $[a_2, b_2] = [a_1, d_1]$; otherwise, we set $[a_2, b_2] = [d_1, b_1]$. Then by induction, we can find a sequence of intervals $[a_n, b_n]$ such that for any $n \in \mathbb{N}$,

$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n], \quad f(a_n) \leq c, \quad f(b_n) \geq c.$$

Then since $\{a_n\}$ and $\{b_n\}$ are monotone and bounded, we have $a_n \to x$ and $b_n \to x$. In fact, there limits are the same as we have $b_n - a_n \to 0$. Since f is continuous, we know

$$f(a_n) \leq c \Rightarrow f(x) \leq c, \quad f(b_n) \geq c \Rightarrow f(x) \geq c.$$

This implies that $f(x) = c$. □

Considering sequences of values for a continuous function that become larger and larger, it is also not hard to prove that continuous functions on closed intervals cannot be unbounded.

Theorem 19.4. *Let I be a bounded closed interval and $f : I \to \mathbb{R}$ be a continuous function. Then f is bounded, and there also exists $a, b \in I$ such that for any $x \in I$,*

$$f(a) \leq f(x) \leq f(b).$$

Proof. First, we show that f is bounded. We argue by contrapositive. Suppose for any $n \in \mathbb{N}$, there exists $x_n \in I$ such that

$$|f(x_n)| \geq n.$$

Then consider the sequence $\{x_n\}_{n\in\mathbb{N}}$. The sequence is bounded, so there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to x. Moreover, $I = [c, d]$ is a closed interval. Since $c \leq x_n \leq d$, we know that

$$c \leq x \leq d.$$

Then as f is continuous, $f(x_{n_k}) \to f(x)$. This contradicts the fact that $|f(x_{n_k})| \to \infty$.

Second, we show that there exists $a \in I$ such that for any $x \in I$,

$$f(x) \geq f(a).$$

Consider $y = \inf\{f(x) \mid x \in I\}$. For any $n \in \mathbb{N}$, there exists $a_n \in I$ such that

$$y \leq f(a_n) \leq y + \frac{1}{n}.$$

The sequence is bounded, so there is a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ that converges to a. Moreover, $I = [c, d]$ is a closed interval. Since $c \leq a_n \leq d$, we know that

$$c \leq a \leq d.$$

Then since f is continuous, we know that $f(a) = y$. □

20. Lecture 20: Open Covers and Continuous Functions

In the previous section, we proved that continuous functions on closed intervals are bounded using contrapositive, by considering a sequence whose values of the function becomes larger and larger. However, is it possible to directly prove this result?

Suppose for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $|x - x'| \leq \delta$, we have

$$|f(x) - f(x')| \leq \epsilon.$$

Then since $\delta > 0$ does not depend on $x \in I$, we can divide the interval I into N intervals of length at most δ. Then we can show f is bounded since for $x, x' \in I$,

$$|f(x) - f(x')| \leq N\epsilon.$$

However, we have discussed the difference between the above definition and the definition of continuity. When f is continuous, we only know that for any $\epsilon > 0$ and any $x \in I$, there exists $\delta_x > 0$, such that for any $|x - x'| \leq \delta_x$, we have

$$|f(x) - f(x')| \leq \epsilon.$$

Hence the question is whether we can choose a minimum for $\delta_x > 0$. One way to do so is to say that the whole interval can be covered by finitely many intervals $(x - \delta_x, x + \delta_x)$. This was considered by Heine in 1872 who gave a proof of uniform continuity of continuous functions on an closed interval.

The following result on closed intervals was proved by Borel in his thesis in 1894, while it also appeared in some form in Dirichlet's unpublished lectures in 1852. Similar techniques has appeared already in Heine's result in 1872 for continuous functions. They realized that one can do this for a bounded closed interval.

Theorem 20.1 (Heine–Borel theorem; Borel 1894)**.** *Let I be a bounded closed interval and $\{U_\alpha\}_{\alpha\in A}$ be a collection of open intervals such that (thus $\{U_\alpha\}_{\alpha\in A}$ is called an open cover of I)*

$$I \subset \bigcup_{\alpha\in A} U_\alpha.$$

Then there exists a finite collection $\{U_n\}_{1\le n\le N} \subset \{U_\alpha\}_{\alpha\in A}$ such that ($\{U_n\}_{1\le n\le N}$ is called a finite subcover)

$$I \subset \bigcup_{n=1}^{N} U_n.$$

Proof. Let $I = J_1 \cup K_1$ where J_1, K_1 are closed intervals with length $\operatorname{diam}(J_1), \operatorname{diam}(K_1) = \operatorname{diam}(I)/2$. Then either J_1 or K_1 does not admit a finite sub-cover of $\{U_\alpha\}_{\alpha\in A}$. Let that sub-interval be I_1. Then consider $I_k = J_{k+1} \cup K_{k+1}$ with length $\operatorname{diam}(J_{k+1}), \operatorname{diam}(K_{k+1}) = \operatorname{diam}(I_k)/2$. We know either J_{k+1} or K_{k+1} does not admit a finite sub-cover of $\{U_\alpha\}_{\alpha\in A}$. Let that sub-interval be I_{k+1}. Thus we defined a sequence of closed intervals

$$I \supset I_1 \supset I_2 \supset \cdots \supset I_k \supset$$

such that $\operatorname{diam}(I_k) \to 0$, and any I_k does not admit a finite subcover of $\{U_\alpha\}_{\alpha\in A}$. Let

$$x \in \bigcap_{k=1}^{\infty} I_k.$$

Since $I \subset \bigcup_{\alpha\in A} U_\alpha$, there exists an open interval such that $x \in U_\alpha$. An open interval always has non-zero length. Then as $\operatorname{diam}(I_k) \to 0$, there exists $k \in \mathbb{N}$ such that $I_k \subset U_\alpha$. This contradicts with the assumption. □

Theorem 20.2 (Heine 1872)**.** *Let I be a closed interval and $f : I \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous, meaning that for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $|x - x'| \le \delta$, we have*

$$|f(x) - f(x')| \le \epsilon.$$

Proof. Since f is continuous, for any $\epsilon > 0$ and any $x \in I$, there exists $\delta_x > 0$, such that for any $|x - x'| \le \delta_x$, we have

$$|f(x) - f(x')| \le \epsilon.$$

We know that since for any $x \in I$, $x \in (x - \delta_x, x + \delta_x)$, we have

$$I \subset \bigcup_{x\in I} (x - \delta_x, x + \delta_x).$$

By the Heine–Borel theorem, we can choose a finite cover such that

$$I \subset \bigcup_{n=1}^{N} (x_n - \delta_n, x_n + \delta_n).$$

Then let $\delta > 0$ be the minimum of the distance between any two non-intersecting open intervals. For any $|x - x'| \le \delta$, there exists $1 \le n \le N$ such that $x, x' \in (x_n - \delta_n, x_n + \delta_n)$. Thus $|f(x) - f(x')| \le \epsilon$. □

Theorem 20.3. *Let I be a closed interval and $f : I \to \mathbb{R}$ be a continuous function. Then f is bounded, and f has maximum and minimum in I.*

Proof. By the above theorem, we know that for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $|x - x'| \leq \delta$, we have

$$|f(x) - f(x')| \leq \epsilon.$$

Divide I into N intervals of lengths at most δ. Suppose the endpoints of these intervals are $x_0, x_1, \dots, x_N$. Then for any $x \in I$, suppose $x \in [x_i, x_{i+1}]$. Then

$$|f(x)| \leq |f(x_0)| + |f(x_1) - f(x_0)| + \cdots + |f(x) - f(x_i)| \leq |f(x_0)| + N\epsilon.$$

Hence f is bounded.

We define $M = \sup\{f(x) \mid x \in I\}$ and claim that there exists $a \in I$ such that

$$f(a) = M.$$

Otherwise, we always have $f(x) < M$. Then we can consider the continuous function

$$g(x) = \frac{1}{M - f(x)}.$$

Since g must also be bounded, we can set $L = \sup\{g(x) \mid x \in I\} > 0$. Then

$$g(x) = \frac{1}{M - f(x)} \leq L, \quad f(x) \leq M - \frac{1}{L} < M.$$

This contradicts the assumption that $M = \sup\{f(x) \mid x \in I\}$. □

21. Lecture 21: Countable and Uncountable Sets

We have seen G. Cantor's construction of the real number system using Cauchy sequences, and how it can be used to deduce various properties of sequences and functions.

However, this was not the end of G. Cantor's adventure. After defining the real number system, he moved on to set up the foundations of descriptive set theory and used that to characterize sets of infinite elements.

We start by recalling the notion of an injection and surjection.

Definition 21.1. *Let $f : A \to B$ be a mapping or function. We say f is injective if for any $a, a' \in A$, $a \neq a'$, we have $f(a) \neq f(a')$. f is surjective if for any $b \in B$, there exists $a \in A$ such that $f(a) = b$. We say that f is bijective if it is both injective and surjective.*

We also recall some basic constructions of sets, by taking the union or the intersection:

Definition 21.2. *Let A and B be sets. We define*

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}, \quad A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Let I be a set and suppose for any $\alpha \in I$, there is a set A_α. Then we define

$$\bigcap_{\alpha \in A} A_\alpha = \{x \mid \text{for any } \alpha \in I, x \in A_\alpha\}, \quad \bigcup_{\alpha \in A} A_\alpha = \{x \mid \text{there exists } \alpha \in I, x \in A_\alpha\}.$$

Let $A \subset X$ be a subset. Define the complement to be

$$A^c = \{x \mid x \in X, x \notin A\}.$$

Theorem 21.1. *Let A_α be subsets of X. Then*

$$\Big(\bigcup_{\alpha \in I} A_\alpha\Big)^c = \bigcap_{\alpha \in I} A_\alpha^c.$$

Proof. We know $x \in (\bigcup_{\alpha \in I} A_\alpha)^c$ means $x \notin \bigcup_{\alpha \in I} A_\alpha$, so there does not exist $\alpha \in I$, such that $x \in A_\alpha$. On the other hand, $x \in \bigcap_{\alpha \in I} A_\alpha^c$ means for any $\alpha \in I$, $x \notin A_\alpha$. □

Using the notion of bijection, Cantor explained that we can compare the number of elements in two sets:

Definition 21.3 (Cantor)**.** *We say that two sets A and B have the same cardinality if there exists a bijection $f : A \to B$. In particular,*

1. *we say A is finite if there exists $n \in \mathbb{N}$ such that A and $\{1, 2, \dots, n\}$ have the same cardinality;*
2. *we say that A is infinite if there does not exist $n \in \mathbb{N}$ such that A and $\{1, 2, \dots, n\}$ have the same cardinality;*
3. *we say that A is countable if A and $\mathbb{N}$ have the same cardinality;*
4. *we say that A is uncountable if A is infinite but not countable.*

Here are some examples of a countable set that is not $\mathbb{N}$: The set of all integers $\mathbb{Z}$ is countable. There is a bijection $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$0, 1, -1, 2, -2, 3, -3, \dots, n, -n, \dots$$

The set of even natural numbers $2\mathbb{N} = \{2n \mid n \in \mathbb{N}\}$ is countable, with the bijection

$$0, 2, 4, 6, \dots, 2n, \dots$$

The set of perfect squares $\{n^2 \mid n \in \mathbb{N}\}$ is countable, with the bijection

$$0, 1, 4, 9, \dots, n^2, \dots$$

Remark 21.1. *We can also say that A is infinite if and only if there exists a bijection between A and a proper subset of A (however, the only if part is very non-trivial to prove).*

Theorem 21.2. *Any infinite subset of a countable set A is countable.*

Proof. First, since A is countable, there is a bijection $a : \mathbb{N} \to A$, so we can denote elements in A by terms in the sequence $\{a_n\}_{n\in\mathbb{N}}$. Let $B \subset A$ be an infinite subset. We define a bijection $b : \mathbb{N} \to B$ as follows. Let $n_1 \in \mathbb{N}$ be the smallest natural number such that $a_{n_1} \in B$. Let $n_{k+1} \in \mathbb{N}$ be the smallest natural number greater than n_1 such that $a_{n_{k+1}} \in B$. Suppose there does not exist $n_{k+1} > n_k$ such that $a_{n_{k+1}} \in B$. Then

$$B = \{a_{n_1}, a_{n_2}, \dots, a_{n_k}\}$$

is finite, which is a contradiction. Therefore, we have an infinite sequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of elements in B. This is an injection by construction. Suppose it is not a surjection, there exists $N \in \mathbb{N}$ such that $a_N \in B$ but $N \neq n_k$ for any $k \in \mathbb{N}$. Then since $n_k \geq k$, we know there exists $k \in \mathbb{N}$ such that

$$n_k < N < n_{k+1}.$$

This contradicts with the construction where n_{k+1} is the minimal number greater than n_k such that $a_{n_{k+1}} \in B$. □

Now we will find more countable and uncountable sets that are not so obvious at first glance.

Theorem 21.3. *Let A_n be a countable set for any $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n$ is also a countable set.*

Proof. Since A_n is countable, we can write elements there as a sequence $\{a_{nk}\}_{k\in\mathbb{N}}$ where $a_n : \mathbb{N} \to A_n$ is a bijection. Then we define a sequence $b : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$ by

$$a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, a_{23}, a_{32}, a_{41}, \dots$$

More formally, we define b_k for any $n(n-1)/2 + 1 \leq k \leq n(n+1)/2$ by induction:

$$b_{n(n-1)/2+1} = a_{n,1},\ b_{n(n-1)/2+2} = a_{n-1,2},\ \dots,\ b_{n(n+1)/2} = a_{1,n}.$$

This defines a bijection $\mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$. □

Corollary 21.4. *Let A be a countable set. Then $A^k = \{(a_1, a_2, \ldots, a_k) \mid a_1, \ldots, a_k \in A\}$ is also countable.*

Proof. We prove by induction on $k \in \mathbb{N}$. When $k = 1$, A is countable by assumption. Suppose A^k is countable. Then $A^{k+1} = A \times A^k$. Consider a bijection $f : \mathbb{N} \to A$. Define the set

$$A_n = \{(f_n, a_1, \ldots, a_k) \mid a_1, \ldots, a_k \in A\}.$$

Then the result follows from the theorem above since $A^{k+1} = \bigcup_{n \in \mathbb{N}} A_n$ and each A_n has a bijection with A^k by $A_n \to A^k, (f_n, a_1, \ldots, a_k) \mapsto (a_1, \ldots, a_k)$. □

Corollary 21.5 (Cantor)**.** *The set of rational numbers $\mathbb{Q}$ is countable.*

Proof. Note that $\mathbb{Q}$ is the set of equivalence classes in $\mathbb{Z} \times \mathbb{Z}$. Hence there is an injection $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{Z} \times \mathbb{Z}$ is countable, $\mathbb{Q}$ is also countable. □

Theorem 21.6 (Cantor)**.** *The set of all the sequences $a : \mathbb{N} \to \{0, 1, \ldots, l\}$ is uncountable.*

Proof. Let $\mathrm{Map}(\mathbb{N}, \{0, 1, \ldots, l\}) = \{a \mid a : \mathbb{N} \to \{0, 1, \ldots, l\}\}$. Suppose there is a bijection $a : \mathbb{N} \to \mathrm{Map}(\mathbb{N}, \{0, 1, \ldots, l\})$. We consider $a_n = \{a_{nk}\}_{k \in \mathbb{N}}$. We now define another infinite sequence $b : \mathbb{N} \to \{0, 1, \ldots, l\}$ such that $b \neq a_n$ for any $n \in \mathbb{N}$. Let $b_k \neq a_{kk}$. Then $b \neq a_n$ because the n-th term b_n is different from a_{nn}. □

22. Lecture 22: Metric Space Part I

After discussing some basic set theory, we will start define and study limits and continuity on more complicated sets.

Definition 22.1. *The d-dimensional Euclidean space is $\mathbb{R}^d = \{(x_1, \ldots, x_d) \mid x_1, \ldots, x_d \in \mathbb{R}\}$, with the addition*

$$(x_1, \ldots, x_d) + (y_1, \ldots, y_d) = (x_1 + y_1, \ldots, x_d + y_d)$$

and inner product or dot product

$$(x_1, \ldots, x_d) + (y_1, \ldots, y_d) = \sum_{i=1}^{d} x_i y_i.$$

Define the norm of $\vec{x} = (x_1, \ldots, x_d)$ to be

$$|\vec{x}| = \sqrt{x_1^2 + \cdots + x_d^2}.$$

Similar to the case of the norm on real numbers, we can prove the triangle inequality for the norm on Euclidean spaces:

Theorem 22.1. *Let $\vec{x}, \vec{y} \in \mathbb{R}^d$. Then we have Cauchy–Schwarz inequality*

$$\vec{x} \cdot \vec{y} \leq |\vec{x}| \cdot |\vec{y}|.$$

We also have the triangle inequality

$$|\vec{x} \pm \vec{y}| \leq |\vec{x}| + |\vec{y}|.$$

Proof. Let $\vec{x} = (x_1, \ldots, x_d)$ and $\vec{y} = (y_1, \ldots, y_d)$. It suffices to prove that

$$\left(\sum_{i=1}^{d} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{d} x_i^2\right) \left(\sum_{i=1}^{d} y_i^2\right).$$

We can directly compute it as follows (details are left to the readers):

$$\left(\sum_{i=1}^{d} x_i y_i\right)^2 - \left(\sum_{i=1}^{d} x_i^2\right)\left(\sum_{i=1}^{d} y_i^2\right) = \sum_{i,j=1}^{d} x_i y_i x_j y_j - \sum_{i,j=1}^{d} x_i^2 y_j^2 = \sum_{1\le i<j\le n} -(x_i y_j - x_j y_i)^2.$$

The triangle inequality follows from Cauchy–Schwarz. □

Similar constructions are not restricted to Euclidean spaces. In fact, Fréchet made the following definition in his thesis which extracts the structure we need to understand limits and continuity:

Definition 22.2 (Fréchet, 1902). *A metric space is a set X together with a function $d : X \times X \to \mathbb{R}$ satisfying the following:*

(1) *for any x, y, $d(x, y) = d(y, x)$;*
(2) *for any x, $d(x, x) = 0$, and for $x \neq y$, $d(x, y) > 0$;*
(3) *for any x, y, z, $d(x, y) \le d(x, z) + d(z, x)$.*

Example 22.1. *(1) The Euclidean space $\mathbb{R}^d$ with the standard distance function*

$$d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$$

is a metric space. The triangle inequality is proved above.

(2) The Euclidean space $\mathbb{R}^d$ with the taxi distance function

$$d(\vec{x}, \vec{y}) = \max_{1\le i\le d}\{|x_i - y_i|\}$$

is also a metric space. The triangle inequality can be proved as follows:

$$\begin{aligned}\max_{1\le i\le d}\{|x_i - y_i|\} &= \max_{1\le i\le d}\{|(x_i - z_i) - (y_i - z_i)|\} \\ &\le \max_{1\le i\le d}\{|x_i - z_i| + |y_i - z_i|\} \\ &\le \max_{1\le i\le d}\{|x_i - z_i|\} + \max_{1\le i\le d}\{|z_i - y_i|\}.\end{aligned}$$

Here, in the last step, we use the fact that

$$\max_{1\le i\le d}\{a_i + b_i\} \le \max_{1\le i\le d}\{a_i\} + \max_{1\le i\le d}\{b_i\}$$

since for any $1 \le i \le d$, $a_i + b_i \le \max_{1\le i\le d}\{a_i\} + \max_{1\le i\le d}\{b_i\}$.

(3) Any set X with the distance function

$$d(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y \end{cases}$$

is a metric space, called the discrete metric space. The triangle inequality can be proved as follows: If $x = y$,

$$d(x, y) = 0 = 0 + 0 \le d(x, z) + d(z, y).$$

If $x \neq y$, then for any $z \in X$, either $z \neq x$ or $z \neq y$, so

$$d(x, y) = 1 = 1 + 0 \le d(x, z) + d(z, y).$$

(4) Let I be a bounded closed interval. The set of continuous functions on a closed interval $C(I)$ with the distance

$$d(f, g) = \sup_{x\in I} |f(x) - g(x)| = \max_{x\in I} |f(x) - g(x)|$$

is a metric space (the maximum of a continuous function on a bounded closed interval always exists). This is one of the first examples of metric spaces that cannot be easily visualized geometrically and is one of the main motivations why Fréchet developed the abstract theory of metric spaces.

The triangle inequality can be proved as follows:

$$\begin{aligned}\max_{x\in I}\{|f(x)-g(x)|\} &= \max_{x\in I}\{|(f(x)-h(x))-(g(x)-h(x))|\}\\ &\le \max_{x\in I}\{|f(x)-h(x)|+|g(x)-h(x)|\}\\ &\le \max_{x\in I}\{|f(x)-h(x)|\}+\max_{x\in I}\{|g(x)-h(x)|\}.\end{aligned}$$

Here, in the last step, we use the fact that

$$\sup_{x\in I}\{a(x)+b(x)\}\le \sup_{x\in I}\{a(x)\}+\sup_{x\in I}\{b(x)\}$$

because for any $x\in I$, $a(x)+b(x)\le \sup_{x\in I}\{a(x)\}+\sup_{x\in I}\{b(x)\}$.

23. Lecture 23: Metric Space Part II

On the set of real numbers, we can define open and closed intervals, and they are used to study limits and continuity. Now, we will define open and closed subsets in a general metric space.

Definition 23.1. *Let X be a metric space and $E\subset X$ be a subset. Then*

1. *a neighborhood of x with radius r is the subset $N_r(x)$ of all points $y\in X$ such that $d(x,y)<r$;*
2. *a point $x\in E$ is an interior point of E if there is a neighborhood $N_r(x)\subset E$;*
3. *the subset E is open if any point $x\in E$ is an interior point.*

Definition 23.2. *Let X be a metric space and $E\subset X$ be a subset. Then*

1. *a point $x\in E$ is a limit point of E if for any neighborhood $N_r(x)$, there exists $y\in E$ and $y\in N_r(x)$ with $x\neq y$;*
2. *the subset E is closed if any limit point of E is contained in E.*

Example 23.1. *$N_r(x)=\{y\in X\mid d(x,y)<r\}$ is always open. This is because for any $y\in N_r(x)$, we can set $r'=r-d(x,y)>0$ and then $N_{r'}(y)\subset N_r(x)$ since $d(z,x)\le d(x,y)+d(z,y)<r$ if $d(z,y)<r'$.*

On the other hand, $\overline{N}_r(x)=\{y\in X\mid d(x,y)\le r\}$ is always closed. If y is a limit point of $\overline{N}_r(x)$, then any $N_\epsilon(y)$ contains a point $z\in\overline{N}_r(x)$, so $d(x,y)\le d(x,z)+d(y,z)\le r+\epsilon$. Since $\epsilon>0$ is arbitrary, we know that $d(x,y)\le r$.

Here are some examples of subsets in Euclidean spaces and the properties they satisfy:

Example 23.2. *Let $X=\mathbb{R}^2$. (1) $N_r(0)=\{\vec{x}\mid|\vec{x}|<1\}$ is open but not closed, $\overline{N}_r(0)=\{\vec{x}\mid|\vec{x}|\le 1\}$ is closed but not open. (2) Any finite set is closed but not open. $\{(n,0)\mid n\in\mathbb{N}\}$ is closed but not open. (3) The set $\{(1/n,0)\mid n\in\mathbb{N}\}$ is neither open nor closed. The set $\{(1/n,0)\mid n\in\mathbb{N}\}\cup\{(0,0)\}$ is closed but not open. (4) A line segment $\{(x,0)\mid x\in(a,b)\}$ is neither open nor closed. A line segment with endpoints $\{(x,0)\mid x\in[a,b]\}$ is closed but not open. (5) Finally, $\mathbb{R}^2$ itself is both closed and open.*

Theorem 23.1. *Let X be a metric space. If x is a limit point of E, then every neighborhood $N_r(x)$ contains infinitely many points in E.*

Proof. Suppose there exists a neighborhood $N_r(x)$ that only contains finitely many points $y_1,\dots,y_k\in E$ if $x\notin E$ or $x,y_1,\dots,y_k\in E$ if $x\in E$. Then let $\delta=\min\{d(x,y_1),\dots,d(x,y_k)\}$. Then $N_\delta(x)$ does not contain any $y\in E$ such that $y\neq x$. Contradiction. □

Open and closed subsets are exactly opposite notions in the following sense:

Theorem 23.2. *Let X be a metric space. Then E is an open subset if and only if the complement E^c is closed.*

Proof. First, suppose E is open. We show that E^c is closed. In other words, there does not exist $x \in E$ that is a limit point of E. Since E is open and $x \in E$, there is a neighborhood $N_r(x) \subset E$. This means $N_r(x) \cap E^c = \varnothing$. Hence x is not a limit point of E.

Next, suppose E^c is closed. We show that E is open. In other words, for any $x \in E$, there exists $N_r(x) \subset E$. Since E^c is closed and $x \notin E^c$, x is not a limit point of E, and there is a neighborhood $N_r(x)$ such that $N_r(x) \cap E^c = \varnothing$. This means $N_r(x) \subset E$. □

Theorem 23.3. *(1) For any collection $\{G_\alpha\}_{\alpha\in A}$ of open sets, $\bigcup_{\alpha\in A} G_\alpha$ is open.*
(2) For any collection $\{F_\alpha\}_{\alpha\in A}$ of closed sets, $\bigcap_{\alpha\in A} F_\alpha$ is closed.
(3) For any finite collection $G_1, \ldots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.
(4) For any finite collection $F_1, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

Proof. We only prove (1) and (3). (1) For any $x \in \bigcup_{\alpha\in A} G_\alpha$, there exists $\alpha \in A$ such that $x \in G_\alpha$. Since G_α is open, there exists a neighborhood $N_r(x) \subset G_\alpha \subset \bigcup_{\alpha\in A} G_\alpha$. (3) For any $x \in \bigcap_{i=1}^n G_i$, we have $x \in G_i$ for any $1 \le i \le n$. Since G_i is open, we know there exists $r_i > 0$ such that $N_{r_i}(x) \subset G_i$. Let $r = \min\{r_1, \ldots, r_n\}$. Then $N_r(x) \subset G_i$ for any $1 \le i \le n$, so $N_r(x) \subset \bigcap_{i=1}^n G_i$. □

For any subset, by taking the interior points, we will get an open subset; and by taking the limit points, we will get a closed subset.

Definition 23.3. *Let X be a metric space and E be a subset. Let E° be the set of interior points of E. Let E' be the set of limit points of E. Then the closure of E is by definition $\overline{E} = E \cup E'$.*

Theorem 23.4. *Let X be a metric space and E be a subset. Then*
(1) *$\overline{E}$ is closed;*
(2) *for any closed subset $F \supset E$, we have $F \supset \overline{E}$.*

Similarly, we have
(1) *E° is open;*
(2) *for any open subset $G \subset E$, we have $G \subset E^\circ$.*

Proof. We only prove the property for the closure.

(1) Suppose x is a limit point of $\overline{E}$. We will show that $x \in \overline{E}$. Suppose any neighborhood $N_r(x)$ contains a point $y_r \in \overline{E}$. When $y_r \in E$, this means $N_r(x)$ contains a point in E. When $y_r \in E'$, this means any $N_r(y_r)$ contains a point $z_r \in E$. By the triangle inequality, we know any $N_{2r}(x)$ contains a point in E. Hence x is a limit point of E.

(2) Suppose F is a closed set and $E \subset F$. Then F contains all its limit points, and in particular it contains limit points of E. □

24. Lecture 24: Compact Subsets

With the definition of open and closed subsets, we can study limits and continuity in metric spaces, following Fréchét.

Definition 24.1. *Let X be a metric space and $E \subset X$ be a subset. Then E is bounded if $\operatorname{diam}(E) = \sup_{x,y\in E} d(x,y) < \infty$.*

Now we define limits of sequences in metric spaces, and generalize the notion of completeness:

Definition 24.2. *Let X be a metric space and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. Then $\{a_n\}_{n\in\mathbb{N}}$ converges to a if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $d(a_n, a) < \epsilon$.*

Definition 24.3. *Let X be a metric space. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$, $d(a_n, a_m) < \epsilon$. A metric space is complete if any Cauchy sequence converges.*

In the previous lectures, we have seen that two key property about limits and continuity is sequential limits and open covers. Both concepts generalize to metric spaces. They are called compactness.

The first notion in terms of sequences and limit points was introduced by Fréchét and then Hausdorff, while the second version in terms of open covers were introduced by Urysohn and Alexandroff at around the same time (who in fact have correpondence with Fréchét).

Definition 24.4 (Fréchét, Hausdorff)**.** *Let X be a metric space. A subset $K \subset X$ is called sequentially compact if any sequence $\{a_n\}$ in K has a convergent subsequence.*

Definition 24.5 (Urysohn, Alexandroff)**.** *Let X be a metric space. A subset $K \subset X$ is called compact if for any open cover $\{U_\alpha\}_{\alpha\in I}$ such that $K \subset \bigcup_{\alpha\in I} U_\alpha$, there is a finite subcover $\{U_n\}_{1\leq n\leq N} \subset \{U_\alpha\}_{\alpha\in I}$ such that $K \subset \bigcup_{n=1}^{N} U_n$.*

These two properties are equivalent on metric spaces. However, the proof is not easy, so we will skip it for now:

Theorem 24.1. *Let X be a metric space. Then a subset $K \subset X$ is compact if and only if it is sequentially compact.*

Remark 24.1. *One can prove that any sequentially compact subset must include all the limit points and hence is closed. Therefore, any compact subset in a metric space is closed.*

Here are some basic properties of compact sets:

Theorem 24.2. *Closed subsets of compact sets are compact.*

Proof. Let $K \subset X$ be a compact set and $F \subset K$ be closed. Consider any open cover $\{U_\alpha\}_{\alpha\in I}$ of F. We will show that there exists a finite open cover. In fact, we enlarge the open cover by considering $\{U_\alpha\}_{\alpha\in I} \cup \{F^c\}$, which is an open cover of K. Since K is compact, there is a finite open sub-cover that covers K and in particular F. However, $F^c \cap F = \varnothing$. This means the finite open sub-cover, after possibly removing F^c, gives a sub-cover of F. □

Compactness implies that any infinite decreasing sequence of compact sets have non-empty intersections. This was Fréchét's original definition of compactness in his thesis.

Theorem 24.3. *Let X be a metric space and $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of compact subsets with $K_n \supset K_{n+1}$ for any $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} K_n$ is non-empty.*

Proof. Consider $U_n = X \setminus K_n$. Since K_n is compact and hence closed, U_n is open. Suppose $\bigcap_{n=1}^{\infty} K_n = \varnothing$. Then $\bigcup_{n=1}^{\infty} U_n = X$, so $\{U_n\}_{n\in\mathbb{N}}$ gives an open cover of X and in particular the compact set K_1. Hence there is a finite cover by $\{U_{n_i}\}_{1\leq i\leq k}$ of the compact set K_1. However, let $N = \max\{n_1, \ldots, n_k\}$. Then we know that $K_N \cap \bigcup_{i=1}^{k} U_{n_i} = \varnothing$. This contradicts the assumption that $K_N \subset K_1 \subset \bigcup_{i=1}^{k} U_{n_i}$. □

25. Lecture 25: Compact Sets in Euclidean Spaces

On the set of real numbers $\mathbb{R}$, we have seen that any closed bounded subset is compact and sequentially compact, by the result of Heine–Borel and Bolzano–Weierstrass. This result does not hold for general metric spaces, but can be easily generalized to Euclidean spaces $\mathbb{R}^d$, with a similar proof.

Theorem 25.1 (Heine–Borel, Bolzano–Weierstrass)**.** *Let $K \subset \mathbb{R}^d$. Then K is compact if and only if K is closed and bounded.*

The main technical input is the following theorem on nested compact sets in $\mathbb{R}^d$:

Theorem 25.2. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of cubes in $\mathbb{R}^k$ of the form $I_n = [a_{n1}, b_{n1}] \times \cdots \times [a_{nk}, b_{nk}] \subset \mathbb{R}^k$. Then $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.*

Proof. We consider the sequence of points $a_n = (a_{n1}, a_{n2}, \dots, a_{nk}) \in \mathbb{R}^k$. Since $\{a_{nj}\}_{n\in\mathbb{N}}$ is monotone and bounded, we know that $a_{nj} \to a_j$. Therefore, $a_n = (a_{n1}, a_{n2}, \dots, a_{nk}) \to (c_1, c_2, \dots, c_k)$. We know that $a_{nj} \leq b_{nj}$, so $a_{nj} \leq c_j \leq b_{nj}$ and thus $c \in \bigcap_{n=1}^{\infty} I_n$. □

Proof of Theorem 25.1. Let $K \subset \mathbb{R}^k$ be a bounded closed subset and $\{U_\alpha\}_{\alpha\in A}$ be an open cover of K. Suppose $\{U_\alpha\}_{\alpha\in A}$ does not have a finite subcover. We will conclude a contradiction. We know there exists $M \in \mathbb{R}$ such that $K \subset [-M, M]^k$. We divide the cube $I_1 = [-M, M]^k$ into 2^k pieces. At least one of them $I_2 \subseteq I_1$ cannot be covered by a finite collection in $\{U_\alpha\}_{\alpha\in A}$. By induction, this defines a sequence of cubes $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \dots$, each I_n cannot be covered by a finite collection in $\{U_\alpha\}_{\alpha\in A}$. Moreover, the diameter $\mathrm{diam}(I_n) \to 0$. Suppose $c \in \bigcap_{n=1}^{\infty} I_n$. We know there exists U_α such that $c \in U_\alpha$. Since U_α is open, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U_\alpha$. Then there exists $n \in \mathbb{N}$ such that $\mathrm{diam}(I_n) \leq \epsilon/2\sqrt{k}$. Then $I_n \subseteq U_\alpha$. This is a contradiction. □

At the end of the section, we remark that the Heine–Borel theorem admits generalizations to certain metric spaces as well, but we need some conditions.

First, the metric space should be complete such that we can take limits of Cauchy sequences. Second, boundedness is not enough; for example, on a discrete metric space, any subset is bounded, but it is easy to find open covers without finite subcovers.

Definition 25.1. *Let X be a metric spaces. A subset $A \subset X$ is totally bounded if for any $\epsilon > 0$ there exists an open cover of A by small balls of radius ϵ.*

Theorem 25.3 (Generalized Heine–Borel)**.** *Let X be a complete metric space and $K \subset X$. Then K is compact if and only if K is closed and totally bounded.*

26. Lecture 26: Connected Sets and Perfect Sets

One important feature about subsets when studying limits and continuity is connectedness. For instance, we know that continuous functions on $[-1, 1]$ and continuous functions on $[-1, 0) \cup (0, 1]$ behave differently: the latter does not have intermediate value property.

Definition 26.1. *Let X be a metric space. Two subsets A and B are called separate if $\overline{A} \cap B$ and $A \cap \overline{B}$ are empty. Then a subset $E \subset X$ is called a connected set if for E is not the union of two separate subsets.*

Theorem 26.1. *Let $E \subset \mathbb{R}$ be connected. Then E is an interval, i.e. for any $x < y \in E$, if $x < z < y$, then $z \in E$.*

Proof. Suppose there exists $x < y \in E$ and $x < z < y$ such that $z \notin E$. Then we define $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$. Since $z \notin E$, we know $E = A \cup B$. Moreover, using properties of limits, we know that $\overline{A} \subset (-\infty, z]$ and $\overline{B} \subset [z, +\infty)$. This implies that A and B are separate. □

Finally, we discuss the relation between topology and cardinality.

Definition 26.2. *Let X be a metric space. Then a subset $P \subset X$ is called a perfect set if any $x \in P$ is a limit point of P.*

Theorem 26.2. *Let $P \subset \mathbb{R}^k$ be a perfect set. Then P is uncountable. In particular, $\mathbb{R}^k$ is uncountable.*

Proof. Since P has limit points, it is infinite. We will suppose that P is countable and deduce a contradiction. Write $P = \{x_n \mid n \in \mathbb{N}\}$. We define a sequence of compact subsets as follows. Let $K_1 = \overline{B}_1(x_1)$. Then $K_1 \cap P$ is infinite since x_1 is a limit point of P. Suppose we have defined $K_1 \supset \cdots \supset K_n$ such that $K_n \cap P$ is infinite. We will let K_{n+1} be a compact subset in K_n such that $x_n \notin K_{n+1}$ and $K_{n+1} \cap P \neq \varnothing$. Then $K_n \cap P$ is compact since P is closed and K_n is closed and bounded. This means $\bigcap_{n=1}^{\infty} K_n \cap P \neq \varnothing$. However, by construction, we also know that for any $n \in \mathbb{N}$, $x_n \notin \bigcap_{n=1}^{\infty} K_n \cap P$. Contradiction. □

The following construction, due to Cantor, shows the surprising fact that a perfect set may not contain any interval of non-zero length. Let $E_0 = [0, 1]$ and $E_k \subset [0, 1]$ be the union of 2^k intervals with length $1/3^k$ defined by

$$E_k = E_{k-1} \setminus \bigcup_{i=1}^{3^{k-1}-1} \left(\frac{3i+1}{3^k}, \frac{3i+2}{3^k} \right).$$

This is to say, for each interval in E_{k-1}, we divide it into 3 subintervals of the same length and remove the middle interval to obtain E_k. Then the Cantor set is

$$C = \bigcap_{n=1}^{\infty} E_n.$$

First, we know that C is non-empty. Second, we also know that C is perfect. This means that we need to show for any $x \in C$ and any $\epsilon > 0$, $(x - \epsilon, x + \epsilon) \cap C \neq \varnothing$. Suppose $x \in E_n$. Denote by I_n the interval in E_n that contains x. Since $\text{diam}(I_n) = 1/3^n$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $I_n \subset (x - \epsilon, x + \epsilon)$. Then let x_n be the endpoint of I_n. We know that $x_n \in C$. Thus $x_n \in (x - \epsilon, x + \epsilon) \cap C$.

27. Lecture 27: Continuity and Compactness/Connectedness

We have discussed various properties about metric spaces, including open and closedness, compactness and connectedness. We will now illustrate how they are useful in understanding continuity of functions on metric spaces.

Definition 27.1. *Let X and Y be metric spaces and $f : X \to Y$ be a map. Suppose for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $0 < d(x, x_0) < \delta$, we have $d(f(x), y_0) < \epsilon$. Then we say that $f(x) \to y_0$ as $x \to x_0$.*

Definition 27.2. *Let X and Y be metric spaces and $f : X \to Y$ be a map. We say that f is continuous at $x_0 \in X$ if $f(x) \to f(x_0)$ as $x \to x_0$; equivalently, if for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $d(x, x_0) < \delta$, we have $d(f(x), y_0) < \epsilon$.*

As we have seen, the precise values of ϵ and δ are not important in the definition of continuity. The notion of open (and closed) subsets now allow us to characterize continuity without using ϵ and δ at all:

Theorem 27.1. *Let X and Y be metric spaces and $f : X \to Y$ be a map. Then f is continuous if and only if for any open subset $V \subset Y$, $f^{-1}(V) = \{x \in X \mid f(x) \in V\} \subset X$ is also open.*

Proof. We know V is open if and only if for any $y \in V$, there exists $\epsilon > 0$ such that the open ball $B_\epsilon(y) \subset V$, and $f^{-1}(V)$ is open if and only if for any $x \in f^{-1}(V)$, there exists $\delta > 0$ such that the open ball $B_\delta(x) \subset f^{-1}(V)$.

Suppose f is continuous and V is open. Then for any $y \in V$, there exists an open ball $B_\epsilon(y) \subset V$. Let $x \in f^{-1}(V)$ such that $f(x) = y$. Since f is continuous, for $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(y)$. This means that $B_\delta(x) \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is also open. The proof of the other direction is similar. □

The most important properties of continuous functions we learned are the existence of maximum and minimum and the intermediate value theorem. They can be formulated in terms of compactness and connectedness.

Theorem 27.2. *Let X and Y be metric spaces and $f : X \to Y$ be continuous. Suppose K is compact. Then $f(K)$ is also compact.*

Proof. We use the characterization of continuity in terms of open sets. Consider any open cover $\{V_\beta\}_{\beta \in I}$ of $f(K)$ such that $f(K) \subset \bigcup_{\beta \in I} V_\beta$. Then it follows that

$$K \subset f^{-1}\Big(\bigcup_{\beta \in I} V_\beta\Big) = \bigcup_{\beta \in I} f^{-1}(V_\beta),$$

because if $x \in f^{-1}(\bigcup_{\beta \in I} V_\beta)$, this means that $f(x) \in \bigcup_{\beta \in I} V_\beta$; hence for some $\beta \in I$, $f(x) \in V_\beta$ and thus $x \in f^{-1}(V_\beta)$. Since f is continuous, $\{f^{-1}(V_\beta)\}_{\beta \in I}$ is a collection of open sets. Since K is compact, there exists a finite collection $\{f^{-1}(V_j)\}_{1 \le j \le n}$ such that

$$K \subset \bigcup_{j=1}^n f^{-1}(V_j).$$

Since $f^{-1}\left(\bigcup_{\beta \in I} V_\beta\right) = \bigcup_{\beta \in I} f^{-1}(V_\beta)$, this then implies that

$$f(K) \subset f\Big(\bigcup_{j=1}^n f^{-1}(V_j)\Big) = f\Big(f^{-1}\Big(\bigcup_{j=1}^n V_j\Big)\Big).$$

In general, we know that $f(f^{-1}(V)) \subset V$ because for any $y \in f(f^{-1}(V))$, there exists $x \in f^{-1}(V)$ such that $y = f(x)$; but then $y = f(x) \in V$. This means that

$$f(K) \subset \bigcup_{j=1}^n V_j.$$

Hence we have found an open subcover. □

Remark 27.1. *Specializing to $f : \mathbb{R} \to \mathbb{R}$, we know closed bounded sets are compact. This means $f([a, b])$ is also closed and bounded, which implies the existence of maximum and minimum.*

It is a useful fact that $f(\bigcup_{\alpha \in I} U_\alpha) = \bigcup_{\alpha \in I} f(U_\alpha)$ and $f^{-1}(\bigcup_{\alpha \in I} V_\alpha) = \bigcup_{\alpha \in I} f^{-1}(V_\alpha)$ (the same result holds for intersections). However, it is not true in general that $f(f^{-1}(V)) = V$. Please find a counterexample yourself.

Theorem 27.3. *Let X and Y be metric spaces and $f : X \to Y$ be continuous. Suppose K is connected. Then $f(K)$ is also connected.*

Proof. Suppose $f(K)$ is disconnected. Write $f(K) = A \cup B$ where A and B are separate. Then note that $K \subset f^{-1}(f(K))$ because for any $x \in K$, $f(x) \in K$ so automatically $x \in f^{-1}(f(K))$. Consider $K \subset f^{-1}(f(K)) = f^{-1}(A) \cup f^{-1}(B)$. We show that $f^{-1}(A)$ and $f^{-1}(B)$ are separate. In fact, we will show that $\overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$. This is because of the following. f is continuous, and hence for the closed set $\overline{A}$, $f^{-1}(\overline{A})$ is also closed. Now we know that $f^{-1}(A) \subset f^{-1}(\overline{A})$, so $\overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$ as the latter is closed. Since $\overline{A} \cap B = \varnothing$, we know $f^{-1}(\overline{A}) \cap f^{-1}(B) = \varnothing$, so in particular $\overline{f^{-1}(A)} \cap f^{-1}(B) = \varnothing$. This shows that $f^{-1}(A)$ and $f^{-1}(B)$ are separate. □

Remark 27.2. *Specializing to $f : \mathbb{R} \to \mathbb{R}$, we know connected sets are all intervals. This means $f([a, b])$ is also an interval, which implies the intermediate value theorem.*

28. Lecture 28: Derivatives of Functions

We can finally introduce derivatives of functions. Unlike continuity which can be defined on general metric spaces, taking derivatives need to be defined on Euclidean spaces.

Definition 28.1. *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. For $x \in I$, define*

$$f'(x) = \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x}.$$

If the above limit exists, we say that f is differentiable at x. We say that $f'(x)$ is the derivative of f at x.

Proposition 28.1. *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Suppose f is differentiable at x. Then f is continuous at x.*

Proof. This is because

$$\lim_{x' \to x} (f(x') - f(x)) = \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x} \lim_{x' \to x} (x' - x) = 0.$$

□

We recall the properties of derivatives that we have learned.

Theorem 28.2. *Let I be an interval. Suppose $f, g : I \to \mathbb{R}$ are differentiable at x. Then $f \pm g$, fg are also differentiable at x:*

1. *$(f \pm g)'(x) = f'(x) \pm g'(x)$;*
2. *$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.*

Proof. We can directly compute

$$\begin{aligned}
(fg)'(x) &= \lim_{x' \to x} \frac{f(x')g(x') - f(x)g(x)}{x' - x} \\
&= \lim_{x' \to x} \frac{f(x')g(x') - f(x)g(x')}{x' - x} + \lim_{x' \to x} \frac{f(x)g(x') - f(x)g(x)}{x' - x} \\
&= \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x} \lim_{x' \to x} g(x') + f(x) \lim_{x' \to x} \frac{g(x') - g(x)}{x' - x} = f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

□

Theorem 28.3. *Let I and J be intervals. Suppose $f : I \to J$ is continuous on I and differentiable at $x \in I$, and $g : J \to \mathbb{R}$ is differentiable at $f(x) \in J$, then $g \circ f(x) = g(f(x))$ is differentiable at x and*

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

Let $y = f(x)$ and $y' = f(x')$. Suppose $y' \neq y$ when $x' \neq x$. Then we can compute that

$$\frac{g(f(x')) - g(f(x))}{x' - x} = \frac{g(y') - g(y)}{y' - y} \frac{f(x') - f(x)}{x' - x}.$$

However, it may not be the case that $y' \neq y$. Therefore, we need to rewrite this identity to avoid $y' - y$ to appear on the denominator.

Proof. Write $g(y') - g(y) = g'(y)(y'-y) + \epsilon_g(y')$ and $f(x') - f(x) = f'(x)(x'-x) + \epsilon_f(x')$. We can compute

$$\begin{aligned} g(f(x')) - g(f(x)) &= g(y') - g(y) = g'(y)(y'-y) + \epsilon(y') \\ &= g'(y)(f(x') - f(x)) + \epsilon_g(y') = g'(y)(f'(x)(x'-x) + \epsilon_f(x')) + \epsilon_g(y') \\ &= g'(y)f'(x)(x'-x) + g'(y)\epsilon_f(x') + \epsilon_g(y'). \end{aligned}$$

When $x' \to x$, we know that $\epsilon_f(x')/(x'-x) \to 0$. Since $y = f(x)$ is continuous, when $x' \to x$, we know that $y' \to y$, and hence $\epsilon_g(y')/(y'-y) \to 0$. This completes the proof. □

Note that not all continuous functions are differentiable; in fact, there even exists continuous functions that are nowhere differentiable.

Example 28.1. *Consider the function*

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right), & x \neq 0, \\ 0 & x = 0. \end{cases}$$

One can show that $f(x)$ is continuous but not differentiable at $x = 0$, because

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \sin\left(\frac{1}{x}\right).$$

29. Lecture 29: Mean Value Theorems

In this lecture, we prove the following important property of differentiable functions: when the function achieves maximum or minimum, the derivative is zero.

Definition 29.1. *Let $f : X \to \mathbb{R}$ be a function. Then we say that f has local maximum at $x \in X$ if there exists a neighborhood $B_\epsilon(x)$ such that for any $x' \in B_\epsilon(x)$, $f(x') \leq f(x)$.*

Theorem 29.1 (Fermat)**.** *Let $f : [a,b] \to \mathbb{R}$ be a function. Suppose f has local maximum at an interior point $x \in (a,b)$ and $f'(x)$ exists. Then $f'(x) = 0$.*

Proof. Since f has local maximum at $x \in I$, we know that there exists $\epsilon > 0$ such that for any $|x - x'| < \epsilon$, $f(x') \leq f(x)$. Now consider $x' > x$. Then we have

$$\frac{f(x') - f(x)}{x' - x} \leq 0.$$

This implies that

$$\lim_{x' \to x^+} \frac{f(x') - f(x)}{x' - x} \leq 0.$$

Consider $x' < x$. Then we have

$$\frac{f(x') - f(x)}{x' - x} \geq 0.$$

This implies that

$$\lim_{x' \to x^-} \frac{f(x') - f(x)}{x' - x} \geq 0.$$

Since $f'(x)$ exists, we know that the above inequalities imply $f'(x) = 0$. □

The following theorem is a generalization of the above result. In fact, if you assume that $f : [a,b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$, then the maximum of f in $[a,b]$ will satisfy that $f'(x) = 0$.

Theorem 29.2 (Lagrange mean value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ differentiable in the interior of (a, b). Then there exists some $x \in (a, b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We consider the function (to reduce the theorem to the precious result)

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then $g(a) = g(b) = 0$ and $g(x)$ is continuous on $[a, b]$ differentiable on (a, b). Since g is continuous on the closed interval $[a, b]$, we can consider the maximum and minimum of g on $[a, b]$. If g has maximum at $x \in (a, b)$, then

$$g'(x) = 0.$$

If g has maximum at $x = a$ or $x = b$, then g must have minimum at $x \in (a, b)$, but then we also have

$$g'(x) = 0.$$

When $g'(x) = 0$, we can conclude that

$$f'(x) - \frac{f(b) - f(a)}{b - a} = 0.$$

This completes the proof. □

Theorem 29.3 (Cauchy mean value theorem)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ differentiable in the interior of (a, b). Then there exists some $x \in (a, b)$ such that*

$$\frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

The following result now follows from the mean value theorem:

Theorem 29.4. *Let $f : I \to \mathbb{R}$ be differentiable.*

(1) *If $f' \geq 0$, then f is monotonely increasing.*
(2) *If $f' \leq 0$, then f is monotonely decreasing.*
(3) *If $f = 0$ on I, then f is constant on I.*

The mean value theorem looks like an intermediate value theorem for the derivative. Indeed, we can show it does imply that the derivative function should satisfy the intermediate value theorem:

Theorem 29.5. *Let $f : [a, b] \to \mathbb{R}$ be differentiable. Suppose $f'(a) \leq c$ and $f'(b) \geq c$. Then there exists $x \in [a, b]$ such that $f'(x) = c$.*

Proof. Let $g(x) = f(x) - cx$. Then $g'(a) \leq 0$ and $g'(b) \geq 0$. This means that $g(x)$ does not obtain minimum at a or b because there exists $\delta > 0$ such that $g(a + \delta) - g(a) \leq 0$ and $g(b - \delta) - g(b) \geq 0$ using the definition of derivatives. Since f is differentiable, it is continuous on $[a, b]$ and hence achieves the minimum at $x \in (a, b)$. Then $f'(x) = 0$. □

Example 29.1. *However, notice that the derivative f' of a differentiable function f is not always continuous (even though it satisfies the intermediate value theorem); for instance, we can take*

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & x \neq 0, \\ 0 & x = 0. \end{cases}$$

30. Lecture 30: L'Hopital's Rule

Theorem 30.1 (L'Hopital's rule, Bernoulli)**.** *Let $f : (a, b) \to \mathbb{R}$ be differentiable. Suppose*

(1) *both $f(x)$ and $g(x) \to 0$ or both $f(x)$ and $g(x) \to \infty$ as $x \to a$,*
(2) $\dfrac{f'(x)}{g'(x)} \to A$ *as $x \to a$.*

Then

$$\frac{f(x)}{g(x)} \to A, \quad x \to a.$$

Proof. Suppose $f(x)$ and $g(x) \to 0$ as $x \to a$. Then one can define a continuous function on $[a, b)$ by

$$f(x) = \begin{cases} f(x), & x \in (a, b), \\ 0, & x = a, \end{cases} \qquad g(x) = \begin{cases} g(x), & x \in (a, b), \\ 0, & x = a. \end{cases}$$

By the mean value theorem, we know that there exists t in between x_0 and x such that

$$\frac{f(x)}{g(x)} = \frac{f'(t)}{g'(t)}.$$

As $x \to x_0$, by squeeze theorem we know $t \to x_0$. This completes the proof.

Suppose $f(x)$ and $g(x) \to \infty$ as $x \to a$. Since $f'(x)/g'(x) \to A$, we may suppose that for $\epsilon > 0$, when $a < x \le a + \delta$, we have

$$\frac{f'(x)}{g'(x)} < A + \epsilon.$$

Then consider $a < x < y < a + \delta$. By mean value theorem, there exists $t \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} \le A + \epsilon.$$

Since $g(x) \to \infty$, we assume that when $a < x \le a + \delta'$, we have

$$g(x) > \frac{1}{\epsilon} f(y), \quad g(x) > -\frac{\epsilon}{A + \epsilon} g(y).$$

Then we can show that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(y)}{g(x)} + \frac{f(y)}{g(x)} = \frac{f(x) - f(y)}{g(x) - g(y)} \frac{g(x) - g(y)}{g(x)} + \frac{f(y)}{g(x)} \le A + 3\epsilon.$$

Similarly, we can prove the inequality on the other side. This will complete the proof. □

31. Lecture 31: Higher Derivatives and Taylor Expansion

We now generalize the mean value theorem into higher order derivatives and prove the Taylor expansion. Let $f : I \to \mathbb{R}$ be a function. We denote by f' its derivative, and write

$$f'' = (f')', \quad f''' = (f'')', \quad f^{(4)} = (f''')', \quad \ldots, \quad f^{(n)} = (f^{(n-1)})'$$

for the higher derivatives of f.

Theorem 31.1. *Suppose $f : [a, b] \to \mathbb{R}$ is n-time differentiable, i.e. $f^{(n-1)} : [a, b] \to \mathbb{R}$ is well-defined, continuous and $f^{(n)} : (a, b) \to \mathbb{R}$ exists. Then there exists $x \in [a, b]$ such that*

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b - a)^k + \frac{f^{(n)}(x)}{n!}(b - a)^n.$$

Proof. The proof is analogous to the mean value theorem. Consider the function

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k - \frac{(t-a)^n}{(b-a)^n}\left(f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k\right).$$

Then one can compute that $g(a) = g'(a) = \cdots = g^{(n-1)}(a) = g(b) = 0$. Since $g(a) = g(b) = 0$, there exists $b_1 \in [a, b]$ such that $g'(b_1) = 0$. Since $g'(a) = g'(b_1) = 0$, there exists $b_2 \in [a, b_1]$ such that $g''(b_2) = 0$. By induction, there exists $x \in [a, b_{n-1}]$ such that $g^{(n)}(x) = 0$. This means that

$$f^{(n)}(x) - \frac{n!}{(b-a)^n}\left(f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k\right) = 0.$$

This finishes the proof. □

You may recall from calculus that for certain functions, we can in fact write down the Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

However, even when f is differentiable infinitely many times, it is not always the case that $f(x)$ can be written as a power series.

First of all, the right hand side has a radius of convergence, and will not converge beyond the radius. For instance, we know the following Taylor series only converges when $|x| < 1$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Second, there is a more serious issue that even when the radius of convergence of the Taylor series is positive, there is no guarantee that $f(x)$ is equal to its Taylor series:

Example 31.1. *Consider the function*

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then one can compute that $f(0) = f'(0) = \cdots = f^{(n)}(0) = 0$. Hence, the Taylor series is equal to zero, and it is not equal to $f(x)$ when $x \neq 0$.

Theorem 31.2. *Suppose $f : [a, b] \to \mathbb{R}$ is infinite-time differentiable, i.e. $f^{(n)} : [a, b] \to \mathbb{R}$ is well-defined, continuous for any $n \in \mathbb{N}$. Suppose that*

$$\lim_{n\to\infty} \frac{1}{n!} \sup_{x\in[a,b]} f^{(n)}(x)(b-a)^n = 0.$$

Then for any $x \in [a, b]$ we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

32. Lecture 32: Riemann Integral Part I

We have finally got to the theory of integration. Of course, smilar to we have seen in the previous lectures, people have been computing integrals long before they knew a rigorous theory of integration. However, in the course of understanding integrations of infinite trigonometric series, people realized that they may encounter very poorly behaved functions.

For instance, we have seen in the previous sections the Dirichlet and Riemann functions, defined by

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases} \quad R(x) = \begin{cases} \frac{1}{q}, & x \in \mathbb{Q}, x = \frac{p}{q},\ gcd(p,q) = 1, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Once these functions come into the theory of calculus, we need to answer the questions whether they are integrable or not.

Definition 32.1 (Riemann 1872)**.** *Let $f : [a,b] \to \mathbb{R}$ be a function. Define a partition P of $[a,b]$ to be a finite set of points*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Write the $\Delta x_i = x_i - x_{i-1}$ and $\Delta P = \max\{\Delta x_i \mid 1 \le i \le n\}$. Let T be a set of points $t_1 < t_2 < \cdots < t_n$ chosen from the partition such that

$$x_{i-1} \le t_i \le x_i, \quad 1 \le i \le n.$$

We define the Riemann sum of the function f on $[a,b]$ with respect to the partition P and the choice of points T to be

$$R(f,P,T) = \sum_{i=1}^{n} f(t_i)\Delta x_i.$$

Let S be a real number. Suppose for any $\epsilon > 0$, there exists $\delta > 0$, such that for any partition P satisfying the condition $\Delta P \le \delta$ and any choice of points T, we have

$$|R(f,P,T) - S| \le \epsilon.$$

Then we say that the Riemann sum of f converges to S, and define the limit S to be the integral of f on $[a,b]$:

$$\int_a^b f(x)dx = \lim_{P,T:\Delta P \to 0} R(f,P,T).$$

We note that the above limit is different from the limits of sequences of limits of functions that we have seen. In fact, in the above limit there are more than one variable: the partition P is changing, meaning that each x_i is changing, and the number of the points n is also changing; the choice of points T is also changing, meaning each t_i is changing, and the number of these points n is also changing. Such feature makes it very hard to directly determine whether the limit exists.

Darboux considered the following approach: while it seems we cannot completely remove the dependence of the Riemann sum on the partition, we can remove the dependence of the Riemann sum on the choice of points T by only considering the supremum and infimum. Now, we follow Darboux's treatment on Riemann integration and study when a function is integrable.

Definition 32.2. *Let $f : [a,b] \to \mathbb{R}$ be a function. Let P be a partition given by $a = x_0 < x_1 < \cdots < x_n = b$. Define*

$$m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}, \quad M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}.$$

Then we define the upper and lower Darboux sum to be

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i, \quad U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

and define the upper and lower integral to be

$$\underline{\int_a^b} f(x)dx = \sup L(f,P), \quad \overline{\int_a^b} f(x)dx = \inf U(f,P).$$

We say that f is Darboux integrable if the upper and lower integrals are the same, and call it the Darboux integral.

33. Lecture 33: Riemann Integral Part II

We will first discuss the notion of Darboux integrable, as it will turn out to be a simpler notion than Riemann integrable.

Proposition 33.1. *Suppose $f : [a,b] \to \mathbb{R}$ is Darboux integrable. Then f is bounded.*

Proof. Suppose f is unbounded. Then for any partition, at least one of the supremums or infimums is infinity, so the Darboux sum will always be infinity. □

We will show that the supremum and infimum of Darboux sums can be achieves by taking finer and finer refinements of the partition. Therefore, the Darboux integral can be viewed as the limit of Darboux sums over the relation of refinements of partitions.

Definition 33.1. *Let P and P' be partitions of $[a,b]$. We say that P' is a refinement of P if $P \subset P'$. Let P_1, P_2 and P' be partitions. We say that P' is a common refinement of P_1 and P_2 if $P_1 \cup P_2 \subset P'$.*

Proposition 33.2. *Let P, P' be partitions and P' be a refinement of P. Then the Darboux sums satisfy*

$$L(f,P) \le L(f,P'), \quad U(f,P') \le U(f,P).$$

Proof. Suppose P' consists of points $a = x_0 < x_1 < \cdots < x_n = b$, and P consists of a subset of points, which we can denote by $a = x_{n_0} < x_{n_1} < \cdots < x_{n_k} = b$, such that $0 = n_0 < n_1 < \cdots < n_k = n$. Then we know that for any $n_i \le j \le j+1 \le n_{i+1}$,

$$\inf\{f(x) \mid x \in [x_{n_i}, x_{n_{i+1}}]\} \le \inf\{f(x) \mid x \in [x_j, x_{j+1}]\}$$

because the interval considered in the former infimum contains the latter. Therefore, for $m_i = \inf\{f(x) \mid x \in [x_{n_i}, x_{n_{i+1}}]\}$ and $m'_j = \inf\{f(x) \mid x \in [x_j, x_{j+1}]\}$,

$$m_i(x_{n_{i+1}} - x_{n_i}) = m_i \sum_{j=n_i}^{n_{i+1}-1} (x_{j+1} - x_j) \le \sum_{j=n_i}^{n_{i+1}-1} m'_j (x_{j+1} - x_j).$$

This completes the proof. □

Proposition 33.3. *Let $f : [a,b] \to \mathbb{R}$ be a function. Then*

$$\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx.$$

Proof. We know that for any partition P,

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i = U(f,P).$$

For any partition P_1 and P_2, consider a common refinement P' of P_1 and P_2. Then we have

$$L(f,P_1) \leq L(f,P') \leq U(f,P') \leq U(f,P_2).$$

Now, we take the supremum for all partitions P_1 (but fix the partition P_2), and get

$$\underline{\int_a^b} f(x)dx \leq U(f,P_2).$$

Then, we take the infimum for all partitions P_2, and get

$$\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx.$$

This completes the proof. □

Theorem 33.4. *Let $f : [a,b] \to \mathbb{R}$ be a function. Then f is Darboux integrable on $[a,b]$ if and only if for any $\epsilon > 0$, there exists a partition P such that*

$$0 \leq U(f,P) - L(f,P) \leq \epsilon.$$

Proof. By the definition of upper/lower bounds and the previous proposition, we know that

$$L(f,P) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq U(f,P).$$

So if for any $\epsilon > 0$, there exists P such that $U(f,P) - L(f,P) \leq \epsilon$, then for any $\epsilon > 0$,

$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq \epsilon.$$

Since the number in the middle is a fixed real number, we can conclude that

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

Conversely, suppose we know that

$$S = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

By the definition of supremum/infimum, we know that for any $\epsilon > 0$, there exists a partition P_1 and a partition P_2 such that

$$S - \epsilon/2 \leq L(f,P_1), \quad U(f,P_2) \leq S + \epsilon/2.$$

Take a common refinement P' of P_1 and P_2. We know that

$$S - \epsilon/2 \leq L(f,P_1) \leq L(f,P') \leq U(f,P') \leq U(f,P_2) \leq S + \epsilon/2.$$

This completes the proof that $U(f,P') - L(f,P') \leq \epsilon$. □

Our next goal is to show that a function is Riemann integrable if and only if it is Darboux integrable. The question is the following: the Riemann integral is the limit over the partitions as we make ΔP smaller, but the Darboux integral is the limit over the partitions as we take refinements. It is not the case that when ΔP becomes smaller, the new partition will become a refinement of the old one.

Theorem 33.5. *Let $f : [a, b] \to \mathbb{R}$ be a function. Then f is Riemann integrable on $[a, b]$ if and only if*
$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx \in (-\infty, +\infty).$$
Moreover, in this case, we have
$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

34. Lecture 34: Riemann Integral Part III

Our next goal is to show that a function is Riemann integrable if and only if its lower and upper Riemann integral are equal. The question is the following: the Riemann integral is the limit over the partitions as we make ΔP smaller, but the Darboux integral is the limit over the partitions as we take refinements. It is not the case that when ΔP becomes smaller, the new partition will become a refinement of the old one.

Theorem 34.1. *Let $f : [a, b] \to \mathbb{R}$ be a function. Then f is Riemann integrable on $[a, b]$ if and only if*
$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx \in (-\infty, +\infty).$$
Moreover, in this case, we have
$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

Proof. Suppose that for $S \in \mathbb{R}$ we have
$$S = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$
This means there exists a partition P such that
$$0 \leq U(f, P) - L(f, P) \leq \epsilon/2.$$
Let $M = \sup\{|f(x)| \mid x \in [a, b]\}$ and n be the number of points in P. Define
$$\delta = \epsilon/4Mn.$$
Now, consider any partition P' such that $\Delta P' \leq \delta$. Let P^* be a common refinement of P and P'. Write
$$P = \{x_0 < x_1 < \cdots < x_n\}, \quad P' = \{x'_0 < x'_1 < \cdots < x'_m\}, \quad P^* = \{x^*_0 < x^*_1 < \cdots < x^*_p\}.$$
First, as P^* is a refinement of P, we know that
$$0 \leq U(f, P^*) - L(f, P^*) \leq \epsilon/2.$$
Next, since P^* is a common refinement of P and P', we can assume that there are n points $x^*_{i_1} < \cdots < x^*_{i_n}$ in P^* that are not in P' coming from P. There are at most $2n$ new intervals in the division containing these points, which we denote by $[x^*_{i_k-1}, x^*_{i_k}]$ and $[x^*_{i_k}, x^*_{i_k+1}]$. There exists a unique $1 \leq j_k \leq m$ such that $[x^*_{i_k-1}, x^*_{i_k}], [x^*_{i_k}, x^*_{i_k+1}] \subset [x'_{j_k-1}, x'_{j_k}]$ and
$$M_{i_k-1} = \sup\{f(x) \mid x \in [x^*_{i_k-1}, x^*_{i_k}]\} \leq \sup\{f(x) \mid x \in [x'_{j_k-1}, x'_{j_k}]\} = M'_{j_k},$$
$$M_{i_k} = \sup\{f(x) \mid x \in [x^*_{i_k}, x^*_{i_k+1}]\} \leq \sup\{f(x) \mid x \in [x'_{j_k-1}, x'_{j_k}]\} = M'_{j_k}.$$

Then since f is bounded, we know that there exists $M > 0$ such that for any $1 \leq j \leq n, 1 \leq i \leq p$, $M'_j, M^*_i \leq M$, and

$$\begin{aligned} U(f,P') - U(f,P^*) &\leq \sum_{k=1}^{n} M_{j'_k}(x'_{j_k} - x'_{j_k-1}) - \sum_{k=1}^{n} M^*_{i_k}(x^*_{i_k} - x^*_{i_k-1}) - \sum_{k=1}^{n} M^*_{i_k+1}(x^*_{i_k+1} - x^*_{i_k}) \\ &= \sum_{k=1}^{n}(M_{j_k} - M^*_{i_k})(x^*_{i_k} - x^*_{i_k-1}) + \sum_{k=1}^{n}(M_{j_k} - M^*_{i_k+1})(x^*_{i_k+1} - x^*_{i_k}) \\ &\leq 2M\sum_{k=1}^{n}(x^*_{i_k} - x^*_{i_k-1}) + 2M\sum_{k=1}^{n}(x^*_{i_k+1} - x^*_{i_k}) \\ &\leq 2M\sum_{k=1}^{n}(x'_{j_k} - x'_{j_k-1}) \leq 2Mn\delta \leq \epsilon/2. \end{aligned}$$

(Similarly, we know that $L(f,P^*) - L(f,P') \leq \epsilon/2$.) This implies that for any choice of variables T' in the partition P', we know that

$$|R(f,P',T') - S| \leq |U(f,P') - S| \leq |U(f,P') - U(f,P^*)| + |U(f,P^*), S| \leq \epsilon.$$

This implies that the function f is Riemann integrable. □

Finally, after clarifying the distinction between different notions of integrations, we can discuss when a function is Riemann integrable.

Theorem 34.2. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable on $[a,b]$.*

Proof. We just need to show that for any $\epsilon > 0$, there exists a partition P such that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \leq \epsilon.$$

Recall that if f is continuous on $[a,b]$, it is uniformly continuous, meaning that for any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - x'| \leq \delta$, we have

$$|f(x) - f(x')| \leq \epsilon/n.$$

Now, choose any partition P such that $\Delta P \leq \delta$. Then

$$M_i - m_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} - \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \leq \epsilon/n.$$

This implies that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \leq \epsilon.$$

which shows that f is Riemann integrable on $[a,b]$. □

However, a function does not need to be continuous for it to be Riemann integrable. For instance, if a function is bounded and only discontinuous at finitely many points, one can imagine it is still integrable because it is integrable on each interval. In fact, a stronger statement holds. This is the Riemann–Lebesgue theorem.

35. Lecture 35: Riemann Integral Part IV

Definition 35.1 (Lebesgue). *Let $E \subseteq \mathbb{R}$ be a subset. Then we say it has measure zero if for any $\epsilon > 0$, there exists a countable collection of interval $I_n = (a_n, b_n)$ such that*

$$E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n), \quad \sum_{n=1}^{\infty}(b_n - a_n) \leq \epsilon.$$

Theorem 35.1. *(1) Any countable set $E \subseteq \mathbb{R}$ has measure zero. (2) The Cantor set $E \subseteq [0,1]$ has measure zero. (3) For any sequence of zero sets $E_n \subseteq \mathbb{R}$, the union $E = \bigcup_{n=1}^\infty E_n$ also has measure zero.*

Proof. (1) Suppose $E = \{x_1, x_2, \ldots, x_n, \ldots\}$. Then for any $\epsilon > 0$, we can consider

$$I_n = \Big(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}}\Big).$$

Then we know that

$$E \subseteq \bigcup_{n=1}^\infty \Big(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}}\Big), \quad \sum_{n=1}^\infty \frac{\epsilon}{2^n} \leq \epsilon.$$

(2) Recall that we can write the Cantor set as

$$E = \bigcap_{k=1}^\infty E_k,$$

where E_n consists of 2^k closed intervals of length $1/3^k$. Then let $E_k = I_1 \cup \cdots \cup I_{2^k}$. Since $(2/3)^k \to 0$, we can choose $k \in \mathbb{N}$ such that $(2/3)^k \leq \epsilon$. Then

$$E \subseteq \bigcup_{n=1}^{2^k} I_n, \quad \sum_{n=1}^{2^k} (b_n - a_n) = \Big(\frac{2}{3}\Big)^k \leq \epsilon.$$

(3) For any $\epsilon > 0$ and $n \in \mathbb{N}$, we choose a countable collection of intervals $I_{n1}, I_{n2}, \ldots, I_{nk}, \ldots$ such that

$$E_n \subseteq \bigcup_{k=1}^\infty I_{nk}, \quad \sum_{k=1}^\infty (b_{nk} - a_{nk}) \leq \frac{\epsilon}{2^n}.$$

Then the collection $\{I_{nk}\}_{n,k\in\mathbb{N}}$ is also a countable collection of intervals, and since rearrangements of absolute convergent series have the same infinite sum, we know that

$$E \subseteq \bigcup_{n,k=1}^\infty I_{nk}, \quad \sum_{n,k=1}^\infty (b_{nk} - a_{nk}) \leq \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \epsilon.$$

This last property is known as the countable additivity property of measure zero sets. □

Theorem 35.2 (Riemann–Lebesgue theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be bounded and only discontinuous at a measure zero set. Then f is Riemann integrable on $[a,b]$.*

Proof. Fix any $\epsilon > 0$. Suppose $|f| \leq M$ and f is only discontinuous at a measure zero set D. Define

$$D_k = \Big\{x \in [a,b] \Big| \limsup_{t\to x} f(x) - \liminf_{t\to x} f(x) \geq \frac{1}{k}\Big\}.$$

Then we know that $D = \bigcup_{k=1}^\infty D_k$ since f is continuous at x if and only if

$$\limsup_{t\to x} f(x) = \liminf_{t\to x} f(x).$$

Since D has measure zero, we know a subset D_k also has measure zero. Consider

$$k \geq 2(b-a)/\epsilon.$$

For any $x \notin D_k$, there exists an interval I_x such that

$$\sup\{f(t) \mid t \in I_x\} - \inf\{f(t) \mid t \in I_x\} \leq \frac{1}{k} \leq \frac{\epsilon}{2(b-a)}.$$

On the other hand, since D_k has measure zero, there exists a sequence intervals $\{J_n\}_{n\in\mathbb{N}} = \{(a_n, b_n)\}_{n\in\mathbb{N}}$ such that

$$D_k \subseteq \bigcup_{n=1}^{\infty} J_n, \quad \sum_{n=1}^{\infty}(b_n - a_n) \le \frac{\epsilon}{4M}.$$

Then $\{I_x\}_{x\notin D_k} \cup \{J_n\}_{n\in\mathbb{N}}$ is an open cover of $[a,b]$, and by Heine–Borel theorem, there exists a finite open cover $\{I_i\}_{i=1}^{M} \cup \{J_j\}_{j=1}^{N}$. Now consider a partition P with K points whose points are exactly given by the endpoints of $\{I_i\}_{i=1}^{M} \cup \{J_j\}_{j=1}^{N}$ (Note that the partition P divide the interval into more pieces since the finite open sets have intersections). Let $A \subseteq \{1, 2, \dots, K\}$ be the subset so that $i \in A$ if $[x_{i-1}, x_i] \subseteq J_j$ for some $1 \le j \le N$. Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{K}(M_i - m_i)\Delta x_i = \sum_{i\in A}(M_i - m_i)\Delta x_i + \sum_{i\notin A}(M_i - m_i)\Delta x_i \\ &\le \sum_{i\in A} 2M\Delta x_i + \sum_{i\notin A}\frac{\epsilon}{2(b-a)}\Delta x_i \le \sum_{n=1}^{\infty} 2M(b_n - a_n) + \sum_{i=1}^{K}\frac{\epsilon}{2(b-a)}\Delta x_i \\ &\le 2M\frac{\epsilon}{4M} + \frac{\epsilon}{2(b-a)}(b-a) = \epsilon. \end{aligned}$$

This shows that f is Riemann integrable. □

36. Lecture 36: Properties of Riemann Integrals

Finally, we can prove the desired properties of Riemann integrals.

Theorem 36.1. *(1) Suppose f and g are Riemann integrable on $[a,b]$. Then $f+g$ and fg are also Riemann integrable, and*

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

(2) Suppose f and g are Riemann integrable on $[a,b]$ and $f \le g$. Then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

(3) Suppose f is Riemann integrable on $[a,b]$ and $c \in [a,b]$. Then f is Riemann integrable on $[a,c]$ and $[c,b]$, and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(4) Suppose f is Riemann integrable on $[a,b]$. Then $|f|$ is Riemann integrable on $[a,b]$, and

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

Theorem 36.2 (Change of variable)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $\phi : [c,d] \to [a,b]$ be monotonely increasing and $\phi' : [c,d] \to \mathbb{R}$ is Riemann integrable. Then $f(\phi(x))\phi'(x)$ is Riemann integrable on $[c,d]$, and*

$$\int_a^b f(y)dy = \int_c^d f(\phi(x))\phi'(x)dx.$$

Proof. Since f is integrable on $[a,b]$, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a,b]$ with $\Delta P \leq \delta$, and any choice of variables T, we have

$$\left| \int_a^b f(y)dy - R(f,P,T) \right| = \left| \int_a^b f(x)dx - \sum_{i=1}^n f(t_i)\Delta y_i \right| \leq \epsilon.$$

Let $y_i = \phi(x_i)$. By the mean value theorem, for any interval $[x_{i-1}, x_i]$, there exists $s_i \in [x_{i-1}, x_i]$ such that

$$\phi'(s_i)\Delta x_i = \phi'(s_i)(x_i - x_{i-1}) = \phi(x_i) - \phi(x_{i-1}) = \Delta y_i.$$

Therefore, by considering $t_i = \phi(s_i)$, we have that

$$\left| \int_a^b f(y)dy - \sum_{i=1}^n f(\phi(s_i))\phi'(s_i)\Delta x_i \right| \leq \epsilon.$$

Since ϕ' is Riemann integrable and f is Riemann integrable, we know that $f(\phi(x))\phi'(x)$ is Riemann integrable, and thi implies that

$$\int_a^b f(y)dy = \int_c^d f(\phi(x))\phi'(x)dx.$$

This completes the proof. □

We end the whole lecture by the Newton–Leibniz theorem, or the fundamental theorem of calculus. The theorem goes back to J. Gregory and I. Barrow, and was finally systematized by I. Newton and G. Leibniz in the early 18th century.

However, for mathematicians in the 18th century, the idea that integration was the inverse of differentiation was a fact that was justified geometrically. It was not until A.-L. Cauchy who formalized integration as a limit of a summation, that people finally realize the necessity to give an algebraic proof. Du Bois-Reymond finally provided such a proof in 1876 (following Cauchy) and named it the fundamental theorem of integral calculus.

Theorem 36.3 (Fundamental theorem of calculus)**.** *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $x \in [a,b]$. Define*

$$F(x) = \int_a^x f(t)dt.$$

Then $F(x)$ is continuous on $[a,b]$. When f is continuous at x, F is differentiable at x and

$$F'(x) = f(x).$$

Proof. First, we prove F is continuous. Suppose $|f| \leq M$. Then it follows that for any $\epsilon > 0$, when $\delta \leq \epsilon/M$, we have

$$|F(x+\delta) - F(x)| = \left| \int_x^{x+\delta} f(t)dt \right| \leq M\delta \leq \epsilon.$$

This shows F is continuous.

Next, we prove that $F'(x) = f(x)$ when f is continuous at x. Since f is continuous at x, we may assume that there exists $\epsilon > 0$ such that for any $|t - x| \leq \delta$,

$$|f(t) - f(x)| \leq \epsilon.$$

We can then compute

$$\left| \frac{F(x+\delta) - F(x)}{\delta} - f(x) \right| = \left| \frac{1}{\delta} \int_x^{x+\delta} f(t)dt - f(x) \right| = \left| \frac{1}{\delta} \int_x^{x+\delta} (f(t) - f(x))dt \right| \leq \epsilon.$$

This finishes the proof. □

Department of Mathematics, University of Southern California.
Email address: wenyuan.li@usc.edu