DIFFERENTIAL TOPOLOGY LECTURES

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1. Lecture 1: Introduction and Preliminaries

Our course is on differentiable manifolds or smooth manifolds, which are topological spaces with sufficiently nice properties that allow one to do calculus.

There are many situations where we would like to do calculus not on $\mathbb{R}^d$ but on more general spaces. For instance, sometimes we need to do differentiation and integrations on curves and surfaces in many circumstances, and sometimes we need to do differentiation and integrations on spaces given by cutting and pasting.

We elaborate more on the second case. For instance, in real analysis, we may want to do calculus on spaces like the circle $S^1 = [0, 1]/\{0 \sim 1\}$, without putting it into Euclidean spaces as $\{(x, y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ all the time. In complex analysis, we may also want to do calculus on spaces like the domain of the n-th root function $f(z) = z^{1/n}$. Note that this function is not well-defined on $\mathbb{C}$. However, it is well defined as a continuous (in fact, smooth) function on the topological space

$$\widetilde{\mathbb{C}}_n = \{(z, k) \mid z \in \mathbb{C}, k \in \mathbb{N}, 1 \leq k \leq n\}/\{(0, i) \sim (0, j)\}$$

by $f(re^{i\theta}, n) = r^{1/n}e^{i\theta/n}$. The latter case motivated B. Riemann to consider the notion which is now known as Riemann surfaces in 1851, which later lead him to introduce the first preliminary definition of manifolds.

In order to do calculus, we would like to assume that the spaces locally just look like standard Euclidean spaces. Recall that a topological space is Hausdorff if for any two distinct points x and y, there exist neighbourhoods U_x and U_y that are also disjoint.

Definition 1.1. *A locally Euclidean space M of dimension d is a Hausdorff topological space M for which each point has a neighborhood homeomorphic to an open subset of the Euclidean space $\mathbb{R}^d$.*

Example 1.1. *(1) Any open subset $U \subseteq \mathbb{R}^d$ is locally Euclidean of dimension d. (2) The circle $S^1 = [0, 1]/\{0 \sim 1\}$, which is homeomorphic to $\{(x, y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$, is locally Euclidean of dimension 1. (3) The cone $C^2_+ = \{(x, y, z) \mid z^2 = x^2 + y^2, z \geq 0\} \subset \mathbb{R}^3$ is locally Euclidean of dimension 2. (4) The sphere*

$$S^d = \{(x_1, x_2, \dots, x_{d+1}) \mid x_1^2 + x_2^2 + \cdots + x_{d+1}^2 = 1\} \subset \mathbb{R}^{d+1}$$

is locally Euclidean of dimension d. Let $n = (0, \dots, 0, 1)$ and $s = (0, \dots, 0, -1)$ be the north and south poles and p_n and p_s be the stereographic projections from n and s:

$$p_n : S^d \setminus \{n\} \to \mathbb{R}^d, \quad p_s : S^d \setminus \{s\} \to \mathbb{R}^d.$$

This gives the local Euclidean structure at any point.

Example 1.2. *(1) The double-sided cone $C^2_\pm = \{(x, y, z) \mid z^2 = x^2 + y^2\} \subset \mathbb{R}^3$ is not locally Euclidean. This can be proved by observing that $C^2_\pm \backslash \{(0, 0, 0)\}$ is disconnected. (2) The one-point union of two circles $S^1 \wedge S^1 = [0, 2]/\{0 \sim 1 \sim 2\}$ is not locally Euclidean. This can be proved by observing that $S^1 \wedge S^1 \backslash \{0\}$ is has three connected components. (3) The half* *space* $\mathbb{R}^2_+ = \{(x_1, x_2) \mid x_2 \geq 0\}$ *is not locally Euclidean. This can be proved by observing that* $\mathbb{R}^2_+ \backslash \{(0,0)\}$ *is simply connected.*

The following example explains when the condition of being a Hausdorff space fails. One reason we would like to exclude this type of examples is that, on these (non-Hausdorff) spaces, we may not be able to find continuous functions that have different values at different points. One of the most effective ways to distinguish points is to look at functions with different values on such points. This is only possible when we consider Hausdorff spaces.

Example 1.3. *Consider the disjoint union of two real lines* $\mathbb{R} \sqcup \mathbb{R}$. *We label a real number* x *in the first* $\mathbb{R}$*-component by* x_{1st} *and a real number* x *in the second* $\mathbb{R}$*-component by* x_{2nd}. *The fat real line* $\mathbb{R}_{\text{fat}} = \mathbb{R} \sqcup \mathbb{R}/\{x_{\text{1st}} \sim x_{\text{2nd}} \mid x \neq 0\}$ *is not locally Euclidean because it is not Hausdorff. In fact, any neighbourhood of* 0_{1st} *and* 0_{2nd} *have non-empty intersections.*

Local homeomorphisms with Euclidean spaces allow us to consider coordinates on the spaces using coordinates on the Euclidean spaces. Coordinates in Euclidean spaces are simply functions $\mathbb{R}^d \to \mathbb{R}$. Therefore, on locally Euclidean spaces, we also define coordinates as some special functions.

Definition 1.2. *Let* M *be a locally Euclidean space. If* x *is a homeomorphism of a connected open set* $U \subseteq M$ *onto an open subset of* $\mathbb{R}^d$, *then* φ *is called a coordinate map. Let* $\pi_i : \mathbb{R}^d \to \mathbb{R}$ *be the standard coordinates. The functions* $x_i = \pi_i \circ x$ *are called the coordinate functions, and the pair* (U, x) *(also denoted by* $(U, x_1, ..., x_d)$*) is called a coordinate system.*

Note that the coordinate function is not always defined on the whole space M. Instead, it is only defined on the open subset U.

Example 1.4. *Consider the circle* $S^1 = [0,1]/\{0 \sim 1\}$. *Then the open subset* $(0,1) \subset S^1$ *is homeomorphic to an open subset in* $\mathbb{R}$. *The coordinate function is given by* $\varphi : (0,1) \to \mathbb{R}$. *It is an exercise that this cannot be extended to a continuous function* $\widetilde{\varphi} : S^1 \to \mathbb{R}$.

2. Lecture 2: Differentiable Manifolds

On locally Euclidean spaces, we now introduce differentiable structures or smooth structures. These will allow us to do calculus on such spaces. The only situation where we know how to characterize differentiability is on $\mathbb{R}^d$. Therefore, we will define smooth structure by requiring certain maps between Euclidean spaces to be smooth. The natural source of such maps come from coordinate systems. The following definition was first introduced by H. Weyl.

Definition 2.1. *A* C^k*-differentiable structure (*$0 \leq k \leq \infty$ *or* ω*, the latter means analytic functions) on a locally Euclidean space* M *is a collection of coordinate systems* $\{(U_\alpha, x_\alpha) \mid \alpha \in A\}$ *satisfying the following properties:*

(1) $\bigcup_{\alpha \in A} U_\alpha = M$;
(2) $(x_\alpha|_{U_\alpha \cap U_\beta}) \circ (x_\beta|_{U_\alpha \cap U_\beta})^{-1}$ *are* C^k*-maps for all* $\alpha, \beta \in A$;
(3) *The collection is maximal with respect to (2); that is, if* (U, x) *is a coordinate system such that* $(x|_{U \cap U_\alpha}) \circ (x_\alpha|_{U \cap U_\alpha})^{-1}$ *and* $(x_\alpha|_{U \cap U_\alpha}) \circ (x|_{U \cap U_\alpha})^{-1}$ *are* C^k*-maps for all* $\alpha \in A$, *then* (U, x) *is also in the collection.*

When $k = \infty$ *or* ω, *we also call this a smooth or analytic structure on* M.

A d-dimensional C^k*-differentiable manifold is a pair* $(M, \mathscr{F})$ *consisting of a d-dimensional, second countable, locally Euclidean space* M *together with a* C^k*-differentiable structure* $\mathscr{F}$. *When* $k = \infty$ *or* ω, *we also call this a smooth or analytic manifold.*

Remark 2.1. *We will not be able to prove this now, but it is true that any C^1-manifold admits a C^k-structure for $k \geq 1$ or $k = \omega$ (by considering a subcollection of coordinate systems). However, it is not true that any C^0-manifold admits a C^1-structure.*

We recall that a space is second countable if it has a countable topological basis (where a topological basis is a collection of open subsets such that all open subsets are forms by unions of them). Later (on Friday) we will explain why we require a manifold to be second countable and Hausdorff.

In the definition of differentiable structures, the last property is to make the definition canonical: whenever two differentiable structures $\mathscr{F} = \{(U_\alpha, x_\alpha) \mid \alpha \in A\}$ and $\mathscr{G} = \{(V_\beta, y_\beta) \mid \beta \in B\}$ are compatible in the sense that all transition maps $x_\alpha \circ y_\beta^{-1}$ and $y_\beta \circ x_\alpha^{-1}$ are differentiable, then $\mathscr{F} = \mathscr{G}$ by property (3).

If $\mathscr{F}_0 = \{(U_\alpha, x_\alpha) \mid \alpha \in A\}$ is any collection of coordinate systems satisfying properties (1) and (2), then we can always find unique differentiable structure $\mathscr{F}$ containing $\mathscr{F}_0$ by

$$\mathscr{F} = \{(U, x) \mid \varphi \circ x_\alpha^{-1}, x_\alpha \circ x^{-1} \text{ are } C^k \text{ for all } \alpha \in A\}.$$

Definition 2.2. *Let M and N be smooth manifolds. We say that $f : M \to \mathbb{R}$ is a smooth function on M (denoted by $C^\infty(M)$ or $C^\infty(M, \mathbb{R})$) if $(f|_{U_\alpha}) \circ x_\alpha^{-1}$ is smooth for each coordinate map $x_\alpha : U_\alpha \to \mathbb{R}^d$. A continuous map $f : M \to N$ is a smooth map (denoted by $C^\infty(M, N)$) if $(y_\beta|_{f(U_\alpha)\cap V_\beta}) \circ (f|_{U_\alpha \cap f^{-1}(V_\beta)}) \circ (x_\alpha|_{U_\alpha \cap f^{-1}(V_\beta)})^{-1}$ is a smooth function for all coordinate maps $x_\alpha : U_\alpha \to \mathbb{R}^d$ and $y_\beta : V_\beta \to \mathbb{R}^c$.*

Lemma 2.1. *Let M and N be smooth manifolds. A continuous map $f : M \to N$ is a smooth map if and only if for any smooth function g on N, the composition $g \circ f$ is a smooth function on M.*

Now we introduce some examples of smooth manifolds. The details are left as exercises.

Example 2.2. *(1) Any open subset in $\mathbb{R}^d$ is a manifold. (2) The general linear group $\mathrm{GL}(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) \neq 0\}$ is a manifold. (3) The product of any two manifolds M and N is still a manifold. (4) The standard sphere*

$$S^d = \{(x_1, x_2, \dots, x_{d+1}) \mid x_1^2 + x_2^2 + \cdots + x_{d+1}^2 = 1\} \subset \mathbb{R}^{d+1}$$

is a manifold. Let $n = (0, \dots, 0, 1)$ and $s = (0, \dots, 0, -1)$ be the north and south poles and p_n and p_s be the stereographic projections from n and s:

$$p_n : S^d \setminus \{n\} \to \mathbb{R}^d, \quad p_s : S^d \setminus \{s\} \to \mathbb{R}^d.$$

Then the maximal collection that contains $\{(S^d \setminus \{n\}, p_n), (S^d \setminus \{s\}, p_s)\}$ gives a smooth structure on S^d. (4) The projective space

$$\mathbb{RP}^d = \{[z_0, z_1, \dots, z_d] \in \mathbb{R}^{d+1} \mid z_0^2 + z_1^2 + \cdots + z_d^2 \neq 0\}/\{[z_0, z_1, \dots, z_d] \sim [\lambda z_0, \lambda z_1, \dots, \lambda z_d]\}$$

is a manifold. One can consider local coordinate systems (U_α, x_α) for $0 \leq \alpha \leq d$ by

$$U_\alpha = \{[z_0, z_1, \dots, z_d] \in \mathbb{R}^{d+1} \mid z_\alpha \neq 0\}, x_\alpha([z_0, z_1, \dots, z_d]) = (z_0/z_\alpha, \dots, z_d/z_\alpha).$$

3. Lecture 3: Partitions of Unity

We mentioned that (one of) the most effective ways to distinguish points is to look at functions with different values on such points. We prove that on manifolds such functions can indeed be constructed. Second countability and Hausdorff will play a major role.

Recall that a collection $\{U_\alpha \mid \alpha \in A\}$ of subsets of M is a cover if $M = \bigcup_{\alpha\in A} U_\alpha$. It is an open cover if each U_α is open. A subcollection which still covers M is called a subcover. A refinement $\{V_\beta \mid \beta \in B\}$ of the cover $\{U_\alpha \mid \alpha \in A\}$ is a cover such that for each $\alpha \in A$ there is an $\beta \in B$ such that $V_\beta \subseteq U_\alpha$. A collection $\{U_\alpha \mid \alpha \in A\}$ of subsets of M is locally finite if whenever $x \in \bigcup_{\alpha \in A} U_\alpha$ there exists a neighborhood W_x of x such that $U_\alpha \cap W_x$ is non-empty for only finitely many $\alpha \in A$. A topological space is paracompact if every open cover has an open locally finite refinement.

Recall that a space is locally compact if each point has at least one compact neighborhood. It is a simple exercise to show that any locally Euclidean space is locally compact.

Lemma 3.1. *Let M be a topological space which is locally compact, Hausdorff, and second countable (manifolds, for example). Then X is paracompact. In fact, each open cover has a countable, locally finite refinement consisting of open sets with compact closures.*

Proof. We decompose M into a countable union of compact subsets and then use (local) compactness to find a (locally) finite refinement. The Hausdorff property is used to show that a compact set is closed.

Given a countable basis of M, since M is local compact, for each point there exists an open subset in the basis that are contained in a compact set. Since M is Hausdorff, we know that such open subsets have compact closures. Consider the countable subcollection $\{U_i\}_{i\in\mathbb{N}}$ of all the open subsets with compact closures. This collection $\{U_i\}_{i\in\mathbb{N}}$ still forms a basis. Let $G_i = \bigcup_{1\le j\le i} U_j$. Then G_i has compact closures, $\overline{G}_i \subseteq G_{i+1}$ and $M = \bigcup_{i\in\mathbb{N}} G_i$. This implies that $\overline{G}_{i+1} \setminus G_i$ are compact subsets.

Now, for any open cover $\{V_\alpha \mid \alpha \in A\}$, we can consider the compact subsets $\{\overline{G}_{i+1} \setminus G_i \mid i \in \mathbb{N}\}$ of M. For each $i \in \mathbb{N}$, there exists a finite subcover. We choose the subcover inductively and this gives a locally finite refinement. □

Lemma 3.2. *Let M be a topological space which is paracompact and Hausdorff. Then M is normal: for any disjoint closed subsets Z_1 and Z_2, there exist open neighbourhoods U_i of Z_i such that U_1 and U_2 are also disjoint.*

Then, using the paracompact and Hausdorff property, we can construct smooth functions that distinguish points. In fact, we can construct partitions of unity:

Definition 3.1. *A partition of unity on M is a collection $\{\varphi_\alpha \mid \alpha \in A\}$ of smooth non-negative functions on M such that*

(1) *The collection of supports $\{\operatorname{supp}\varphi_\alpha \mid \alpha \in A\}$ is locally finite.*
(2) *$\sum_{\alpha\in A} \varphi_\alpha(x) = 1$ for all $x \in M$.*

A partition of unity $\{\varphi_\alpha \mid \alpha \in A\}$ is subordinate to the cover $\{U_\beta \mid \beta \in B\}$ if for any $\beta \in B$, there exists some $\alpha \in A$ such that $\operatorname{supp}\varphi_\alpha \subset U_\beta$.

We remark that for a partition of unity to be subordinate to a cover, one open subset U_α may contain the support of many functions φ_β.

Theorem 3.3 (Existence of Partitions of Unity)**.** *Let M be a differentiable manifold and $\{U_\alpha \mid \alpha \in A\}$ an open cover of M. Then there exists a countable partition of unity with compact supports $\{\varphi_i \mid i \in \mathbb{N}\}$ subordinate to the cover $\{U_\alpha \mid \alpha \in A\}$.*

Lemma 3.4. *There exists a non-negative smooth function f on $\mathbb{R}^d$ which equals 1 on $[-1,1]^d$ and 0 on the complement of $(-2,2)^d$.*

Proof. We define $f(x_1,\dots,x_d) = h(x_1)h(x_2)\dots h(x_d)$, where

$$h(t) = g(2+t)g(2-t), \quad g(t) = \frac{u(t)}{u(t)+u(1-t)}, \quad u(t) = \begin{cases} e^{-1/t}, & t>0, \\ 0, & t \le 0. \end{cases}$$

This satisfies the condition. □

Proof of Theorem. Consider a collection $\{(V_x, \varphi_x) \mid x \in M\}$ of coordinate systems that is a refinement of $\{U_\alpha \mid \alpha \in A\}$. For each (V_x, φ_x), by the lemma above, we can consider a function ρ that is compactly supported in $\varphi_x(V_x)$ and equal to 1 on an open subset $B_x \subset \varphi_x(V_x)$. Then the function $f_x = f \circ \varphi_x$ is compactly supported in V_x and equal to 1 on an open subset $W_x \subset V_x$ (where $W_x = \varphi_x^{-1}(B_x)$). Now the collection $\{W_x \mid x \in M\}$ forms an open cover that refines $\{U_\alpha \mid \alpha \in A\}$. Since M is second countable and paracompact, we can choose a countable locally finite cover $\{W_i \mid i \in \mathbb{N}\}$ that also refines $\{U_\alpha \mid \alpha \in A\}$. We consider

$$\rho_i = \frac{f_i}{\sum_{j\in\mathbb{N}} f_j}.$$

Then $\{\rho_i \mid i \in \mathbb{N}\}$ forms a partition of unity subordinate to $\{U_\alpha \mid \alpha \in A\}$. □

Remark 3.1. *We remark that the above proof also shows that any second countable paracompact Hausdorff space admits a continuous partition of unity.*

Corollary 3.5. *Let M be a differentiable manifold, Z be a closed subset and U be an open neighbourhood of Z. Then there exists a smooth function φ on M such that*

(1) $0 \le \varphi(x) \le 1$;
(2) $\varphi(x) = 1$ *for* $x \in Z$;
(3) $\varphi(x) = 0$ *for* $x \notin U$.

Such functions are called bump functions or cut-off functions.

4. Lecture 4: Tangent Vectors and Differentials

Our goal this week is to define derivatives and differentials locally on a smooth manifold. More precisely, we will define tangent spaces and cotangent spaces at a point. Tangent vectors will give directional derivatives at a point, and cotangent vectors will give differentials of functions at a point.

The difficulty of defining the tangent and cotangent spaces is that manifolds are only topological spaces and there is no vector space structures. However, one natural vector space that arises from a manifold is the vector space of functions $C^\infty(M)$. We will start from that.

We will first try to define cotangent spaces. This is because cotangent vectors are differentials of functions at a point, and differentials, roughly speaking, are simply the first-order approximations of functions.

Definition 4.1. *Let M be a smooth manifold and p be a point on M. Consider the set of smooth functions defined on some (open) neighbourhood of p, namely, the set of pairs (U, f) such that U is an open neighbourhood of p and f is in $C^\infty(U)$. Define an equivalence relation such that $(U, f) \sim (V, g)$ if there exists an open neighbourhood $W \subset U \cap V$ of x such that $f|_W = g|_W$. Define*

$$C_p^\infty = \{(U, f) \mid U \text{ open neighbourhood of } p,\ f \in C^\infty(U)\}/\sim$$

to be the vector space of germs of smooth functions at $p \in M$.

Use partitions of unity, one can prove that germs of smooth functions have a simpler description. (The reason to make the more complicated definitions is that germs make sense even when we do not have partitions of unity. For instance, one can define germs of analytic functions in the above way.)

Lemma 4.1. *Let M be a smooth manifold and p be a point on M. Define an equivalence relation on $C^\infty(M)$ such that $f \sim g$ if there exists an open neighbourhood W of p such that $f|_W = g|_W$. Then $C_p^\infty = C^\infty(M)/\sim$.*

Differentials of functions the space of functions that vanish at the point modulo second and higher order terms. Recall the following simple fact: if two smooth functions f_1 and f_2 vanish at a point x, then $f_1 f_2$ vanish at x up to the second order (by the Leibniz rule). This gives an alternative description of functions that vanish up to the second order.

Definition 4.2. *Let M be a smooth manifold and C_p^∞ is the space of germs of smooth functions at p. Define $m_p \subset C_p^\infty$ to be the subspace of germs of functions $f \in C_p^\infty$ such that $f(p) = 0$ and $m_p^2 \subset C_x^\infty$ to be subspace spanned by germs of functions of the form $f_1 f_2 \in C_x^\infty$ such that $f_1(p) = f_2(p) = 0$. Then the cotangent space at $p \in M$ is*

$$T_p^* M = m_p / m_p^2.$$

Theorem 4.2. $\dim T_p^* M = \dim M$.

Recall that Newton–Leibniz rule implies that for a C^1-differentiable function on $\mathbb{R}^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have

$$f(x) = f(0) + \sum_{i=1}^{d} x_i \int_0^1 (1-t) \frac{\partial f}{\partial x_i}(tx) dt.$$

Doing it once more implies that for a C^2-differentiable function on $\mathbb{R}^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have Taylor's formula with integral remainders

$$f(x) = f(0) + \sum_{i=1}^{d} x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j=1}^{d} x_i x_j \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(tx) dt.$$

Proof of the Theorem. Consider a local coordinate system (U, x) around p such that $x(p) = 0$ in $\mathbb{R}^d$. Then for any germ of function $f \in m_p$ that vanishes at p, we can consider the function $f \circ x^{-1}$ and write

$$f(q) = \sum_{i=1}^{d} x_i(q) \frac{\partial (f \circ x^{-1})}{\partial x_i}(0) + \sum_{i,j=1}^{d} x_i(q) x_j(q) \int_0^1 (1-t) \frac{\partial^2 (f \circ x^{-1})}{\partial x_i \partial x_j}(tx(q)) dt.$$

Since the last term is a linear combination of functions that vanish at p, we have in m_p/m_p^2 that

$$f(q) \equiv \sum_{i=1}^{d} x_i(q) \frac{\partial (f \circ x^{-1})}{\partial x_i}(p).$$

This implies that $\{x_1, \ldots, x_d\}$ spans m_p/m_p^2. Now, we can also show that they are linear independent. This completes the proof. $\square$

Definition 4.3. *Let M be a smooth manifold and p be a point in M. We write the image of a germ of smooth functions f in m_p/m_p^2 as the differential df_p. In particular, for a coordinate system (U, x) around p. We write the image of $x_1, \ldots, x_d$ in m_p/m_p^2 as the differentials $(dx_1)_p, \ldots, (dx_d)_p$.*

Next, we try to define the tangent space. They are, roughly speaking, directional derivatives at a point. Derivatives are operators that act on smooth functions, and we can define them in the following axiomatic way.

Definition 4.4. *A tangent vector v at the point $p \in M$ is a linear derivation on C_p^∞. That is, for all $f_1, f_2 \in C_p^\infty$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, the map $v : C_p^\infty \to \mathbb{R}$ satisfies*

(1) $v(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 v(f_1) + \lambda_2 v(f_2)$;
(2) $v(f_1 f_2) = v(f_1) f_2(p) + f_1(p) v(f_2)$.

The tangent space $T_p M$ at $p \in M$ is the vector space of tangent vectors at $p \in M$.

Lemma 4.3. *T_pM is naturally isomorphic to the linear dual of T_p^*M (that is, all linear functions on T_p^*M).*

Proof. Using Leibniz rule, we know that for any $f \in m_p^2$ and tangent vector $v : C_p^\infty \to \mathbb{R}$, we have $v(f) = 0$. Therefore, a tangent vector induces a linear function $m_p/m_p^2 \to \mathbb{R}$. Conversely, for any linear function $\ell : m_p/m_p^2 \to \mathbb{R}$, we can define a tangent vector by $v_\ell(f) = \ell(f - f(p))$. It satisfies Leibniz rule because

$$\begin{aligned} v_\ell(f_1f_2) &= \ell(f_1f_2 - f_1(p)f_2(p)) = \ell((f_1 - f_1(p))f_2(p) + f_1(f_2 - f_2(p))) \\ &= \ell((f_1 - f_1(p))f_2(p)) + \ell(f_1(p)(f_2 - f_2(p))) = v_\ell(f_1)f_2(p) + f_1(p)v_\ell(f_2). \end{aligned}$$

Here, the second last equality is because $(f_1 - f_1(p))(f_2 - f_2(p)) \in m_p^2$, which implies that

$$\ell(f_1(p)(f_2 - f_2(p))) = \ell(f_1(f_2 - f_2(p))).$$

This shows that there is a natural bijection. □

Definition 4.5. *Let M be a smooth manifold and p be a point in M. For a local coordinate system (U, x) around p, we define the dual basis of $(dx_1)_p, \dots, (dx_d)_p \in T_p^*M$ to be $(\partial/\partial x_1)_p, \dots, (\partial/\partial x_d)_p \in T_pM$. That means*

$$dx_i\big(\frac{\partial}{\partial x_j}\big)_p = \big(\frac{\partial x_i}{\partial x_j}\big)_p = \delta_{ij}.$$

5. Lecture 5: Pull Back and Push Forward

We can explain why we call the linear derivatives tangent vectors in a geometric way. Intuitively, tangent vectors should be the direction vectors of smooth curves, in other words, they are first order approximations of smooth curves. We have seen that one can define first order term using equivalence relations. Here, we do the same thing.

The difficulty is to measure equality up to the first order. This means we need to consider smooth functions and take the first derivatives. This leads to the following definition:

Definition 5.1. *Let M be a smooth manifold and p be a point in M. Consider the set of smooth curves defines on some neighbourhood of p, namely, the set of pairs (U, γ) where U is an open neighbourhood of x and $\gamma : (-1, 1) \to U$ is a smooth map such that $\gamma(0) = p$. Define the equivalence relation such that $(U, \gamma) \sim (V, \delta)$ if there exists an open neighbourhood $W \subset U \cap V$ of x such that for any smooth function $f \in C^\infty(W)$,*

$$\Big(\frac{d}{dt}(f \circ \gamma(t))\Big)_{t=0} = \Big(\frac{d}{dt}(f \circ \delta(t))\Big)_{t=0}.$$

Define $T'_pM = \{(U, \gamma) \mid U$ an open neighbourhood of x, $\gamma \in C^\infty((-1, 1), U), \gamma(0) = p\}$.

It is not clear that this definition gives a vector space, but it turns out that it does give the same tangent space as we have in the previous lecture.

Lemma 5.1. *There is a natural bijection between T'_pM and T_pM.*

Now we explain how to compute derivatives and differentials of smooth functions on a manifold.

Remark 5.1. *(1) For a manifold M and a coordinate system (U, φ) around p, a tangent/cotangent vector is of the form*

$$v = \sum_{i=1}^d v_i\big(\frac{\partial}{\partial x_i}\big)_p, \quad \alpha = \sum_{i=1}^d \alpha_i(dx_i)_p.$$

(2) When $M = \mathbb{R}^d$ and x is the standard coordinate system, the differential of a smooth function $f \in C^\infty(\mathbb{R}^d)$ at $p \in \mathbb{R}^d$ is simply

$$df_p = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(p)(dx_i)_p.$$

More generally, for M with a coordinate system (U, x) around p, the differential of $f \in C^\infty(M)$ at $p \in M$ is simply

$$df_p = \sum_{i=1}^d \frac{\partial(f \circ x^{-1})}{\partial x_i}(x(p))(dx_i)_p.$$

(3) For two coordinate systems (U, x) and (V, y) around p, we have the coordinate change formula that

$$\frac{\partial}{\partial y_j} = \sum_{i=1}^d \frac{\partial x_i}{\partial y_j}\frac{\partial}{\partial x_i}, \quad dy_j = \sum_{i=1}^d \frac{\partial y_j}{\partial x_i} dx_i.$$

Here the matrix $(\partial y_j/\partial x_i)_{i,j=1}^n$ is the Jacobian matrix of the smooth function $y \circ x^{-1}$.

Given a smooth map, it induces maps between tangent spaces and cotangent spaces. This is a generalization of change of coordinates in Euclidean spaces.

Definition 5.2. *Let $\varphi : M \to N$ be a smooth map and $p \in M$. Then the pull back map of φ is a linear map $\varphi^* : T^*_{\varphi(p)}N \to T^*_p M$ defined by*

$$\varphi^*(df)_{\varphi(p)} = d(f \circ \varphi)_p.$$

The push forward map or differential of φ is a linear map $\varphi_* : T_p M \to T_{\varphi(p)}N$ (also denoted by $d\varphi$) defined by

$$\varphi_* v(f) = v(f \circ \varphi).$$

In particular, for $\varphi : M \to \mathbb{R}$, we have $\varphi_* v = d\varphi(v)$.

Remark 5.2. *(1) For coordinate systems (U, x) and (V, y) around p and $\varphi(p)$, we can write $\varphi_j = y_j \circ \varphi$ for the j-th component of φ and get*

$$\varphi_*(\frac{\partial}{\partial x_i}) = d\varphi(\frac{\partial}{\partial x_i}) = \sum_{j=1}^d \frac{\partial \varphi_j}{\partial x_i}\frac{\partial}{\partial y_j}.$$

Here the matrix $(\partial\varphi_j/\partial x_i)_{i,j}$ is the Jacobian matrix of the smooth map φ.

(2) For a smooth curve $\gamma : (-1, 1) \to M$, considering the tangent vector d/dt on $\mathbb{R}$ with respect to the standard coordinate system, we write

$$\gamma_*(\frac{d}{dt}) = d\gamma(\frac{d}{dt}) = \gamma'(t).$$

One can show that this is the tangent vector defined at the beginning of the lecture as

$$\gamma_*(\frac{d}{dt})(f) = \frac{d}{dt}(f \circ \gamma).$$

(3) For two maps $\varphi : M \to N$ and $\psi : N \to P$, we have chain rules that $(\psi\circ\varphi)_* = \psi_* \circ \varphi_*$ or $d(\psi \circ \varphi) = d\psi \circ d\varphi$, and $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. The latter follows from the definition of pull back maps and compositions of functions.

6. Lecture 6: Submanifolds and Inverse Function Theorem

We define submanifolds and give more examples of manifolds and submanifolds.

Definition 6.1. *Let $\varphi : M \to N$ be a smooth map.*

1. *φ is an immersion if $d\varphi_p$ is injective for all $p \in M$.*
2. *φ is a submersion if $d\varphi_p$ is surjective for all $p \in M$.*
3. *φ is an injective immersion if it is an immersion and injective.*
4. *φ is an embedding if it is an immersion and homeomorphism onto the image.*
5. *φ is a diffeomorphism if it is an embedding and surjective.*

Example 6.1. *A curve $\gamma : \mathbb{R} \to M$ is an immersion if $\gamma'(t) = \gamma_*(d/dt) \neq 0$. When $\gamma(s) \neq \gamma(t)$ whenever $s \neq t$, the curve is an injective immersion. The curve $\gamma : \mathbb{R} \to T^2$ is defined by $\gamma(t) = (t, \alpha t)$ for some $\alpha \notin \mathbb{Q}$ is an injective immersion but not an embedding.*

We recall the inverse function theorem in Euclidean spaces. Let $U \subset \mathbb{R}^d$ be an open subset containing 0 and $\varphi : U \to \mathbb{R}^d$ be a smooth map such that $\varphi(0) = 0$. Suppose at $0 \in U$, the Jacobian

$$J(\varphi)_0 = \Big(\frac{\partial \varphi_i}{\partial x_j}\Big)_{1 \leq i,j \leq d}$$

is invertible. Then the inverse function theorem states that there exists a smaller open subset $V \subset U$ an inverse function $\psi : \varphi(V) \to V$ such that

$$\varphi \circ \psi = \mathrm{id}, \quad \psi \circ \varphi = \mathrm{id}.$$

The outline of the proof is as follows:

First, using the contraction mapping principle, We show that the linear map $J(\varphi)_0$ is injective implies that φ is also injective in a neighbourhood of 0. Since $J(\varphi)_0$ is injective, we may assume that $|J(\varphi)_0(p) - J(\varphi)_0(q)| \geq C|p - q|$. For $\epsilon(p) = \varphi(p) - J(\varphi)_0(p)$, there is a small neighbourhood of 0 so that

$$|\epsilon(p) - \epsilon(q)| \leq \frac{C}{2}|p - q|.$$

Then we have the inequality which shows that φ is injective

$$|p - q| \leq \frac{C}{2}|\varphi(p) - \varphi(q)|.$$

Using the contraction mapping principle, we can show that φ is surjective on a neighbourhood of $\varphi(0)$ because $F : p \mapsto q - \epsilon(p)$ is a contraction mapping and therefore has a fixed point. Then, we prove that the inverse ψ of φ is continuously differentiable using the definition.

This theorem can be generalized to smooth manifolds.

Theorem 6.1 (Inverse Function Theorem)**.** *Let $\varphi : M \to N$ be a smooth map and $\varphi_* : T_pM \to T_{\varphi(p)}N$ is an isomorphism for some $p \in M$. Then there exists an open neighbourhood U of p such that $\varphi : U \to \varphi(U)$ is a diffeomorphism.*

Proof. Consider local coordinate systems (U, x) and (V, y) around p and $\varphi(p)$. Under the basis $dx_1, \dots, dx_d$ and $dy_1, \dots, dy_d$, the pull back map $\varphi^* : T^*_{\varphi(p)}N \to T^*_pM$ is represented by the matrix

$$\Big(\frac{\partial \varphi_i}{\partial x_j}\Big)_{1 \leq i,j \leq d}.$$

Since φ_* is an isomorphism, we know that φ^* is also an isomorphism and the matrix is invertible. This is now reduced to the case of Euclidean spaces. □

7. Lecture 7: Inverse Function Theorem and Coordinates

The inverse function theorem provides an effective way to construct local coordinate systems.

Definition 7.1. *A set $y_1, \dots, y_d$ of smooth functions defined on some neighborhood of $p \in M$ is called an independent set at p if $dy_1, \dots, dy_d$ are linear independent in T_p^*M.*

Corollary 7.1. *Let M be a d-dimensional manifold and $y_1, \dots, y_d$ be independent functions at p. Then $y_1, \dots, y_d$ forms a coordinate system in a neighbourhood U of p.*

Proof. Consider the smooth map $y : M \to \mathbb{R}^d$. Apply the inverse function theorem, we know that y is a diffeomorphism in a neighbourhood U of p. Consider any coordinate system $x : V \to \mathbb{R}^d$. Then $(y|_{U\cap V}) \circ (x|_{U\cap V})^{-1}$ and $(x|_{U\cap V}) \circ (y|_{U\cap V})^{-1}$ are smooth maps. This implies that (U, y) is also a coordinate system. □

Corollary 7.2. *Let M be a d-dimensional manifold and $x_1, \dots, x_l$ be independent functions at p. Then $x_1, \dots, x_l$ form part of a coordinate system in a neighbourhood U of p.*

Proof. Consider a local coordinate system (V, y) around p. Then $dx_1, \dots, dx_l, dy_1, \dots, dy_d$ span T_p^*M. Since $dx_1, \dots, dx_l$ are linear independent, we can choose a basis of T_p^*M given by $dx_1, \dots, dx_l, dy_{i_1}, \dots, dy_{i_{d-l}}$. Then apply the previous corollary. □

Corollary 7.3. *Let $\varphi : M \to N$ be a submersion at p and $x_1, \dots, x_d$ be a coordinate system around $\varphi(p)$. Then $x_1 \circ \varphi, \dots, x_d \circ \varphi$ form part of a coordinate system in a neighbourhood U of p.*

Proof. Since the push forward φ_* is surjective, we know that the pull back φ^* is injective. Since $dx_1, \dots, dx_d$ form a basis in $T_{\varphi(p)}^*N$, we conclude that $d(x_1 \circ \varphi), \dots, d(x_d \circ \varphi)$ are linear independent in T_p^*M. Then apply the previous corollary. □

Corollary 7.4. *Let $\varphi : M \to N$ be an immersion at p and $x_1, \dots, x_d$ be a coordinate system around $\varphi(p)$. Then a subcollection of $x_1 \circ \varphi, \dots, x_d \circ \varphi$ forms a coordinate system in a neighbourhood U of p.*

Finally, we show that all immersions have standard local forms. Namely, they are slices in local coordinate systems:

$$\{p \in U \mid x_i(p) = c_i, l+1 \le i \le d\}.$$

Proposition 7.5. *Let $\varphi : M \to N$ be an immersion. Then for any $p \in M$, there exists a coordinate system (V, x) centered at $\varphi(p)$ and a neighbourhood U of p such that $\varphi|_U$ is injective and $\varphi(U)$ is a slice in (V, x).*

Proof. Consider a coordinate system (W, y) centered at $\varphi(p)$. By the corollary, a subcollection of $y_1 \circ \varphi, \dots, y_d \circ \varphi$ forms a coordinate system in a neighbourhood U of p. We denote that by $z_1, \dots, z_l$. Then on the open neighbourhood $V = W \cap y^{-1}(\mathbb{R}^l \times \varphi(U))$, we can consider the projection $\pi : \mathbb{R}^d \to \mathbb{R}^l$ and define

$$x_i = y_i,\ 1 \le i \le l; \quad x_i = y_i - y_i \circ \varphi \circ z^{-1} \circ \pi \circ y,\ l+1 \le i \le d.$$

Suppose the linear homomorphism $d\varphi : T_pM \hookrightarrow T_{\varphi(p)}N$ is given by $dy_i = dz_i$ for $1 \le i \le l$ and $dy_i = \sum_{j=1}^{l} a_{ij} dz_i$ for $l+1 \le i \le d$. One can show that this defines a coordinate system by computing the differentials:

$$dx_i = dy_i,\ 1 \le i \le l, \quad dx_i = dy_i - \sum_{j=1}^{l} a_{ij} dy_j,\ l+1 \le i \le d.$$

Moreover, for $q \in \varphi(N)$, we always have $x_i(q) = y_i(q) - y_i \circ \varphi \circ z^{-1} \circ \pi \circ y(q) = 0$. This completes the proof. □

We remark that the above proposition is local in M but not local in N. Namely, we need to fix an open set U in M and then find an open set V in N so that $\varphi(U) \subset V$. It is not true that we can directly find an open set V in N so that $\varphi(M) \cap V$ is a slice, unless φ is an embedding.

8. Lecture 8: Implicit Function Theorem

Many interesting subspaces in Euclidean spaces are cut out by certain (system of) equations. In this lecture, we provide a general criterion on when equations of smooth functions cut out a submanifold. This will provide a large number of examples.

Definition 8.1. *Let $\varphi : M \to N$ be a smooth map. Then $q \in N$ is a regular value of φ if for any $p \in \varphi^{-1}(q)$, $d\varphi_p$ is a submersion.*

Theorem 8.1 (Implicit Function Theorem)**.** *Let $\varphi : M \to N$ be a smooth map and q is a regular value of φ. Then $\varphi^{-1}(q)$ has a (unique) manifold structure such that the inclusion $\varphi^{-1}(q) \hookrightarrow M$ defines an embedded submanifold whose dimension is* $\dim M - \dim N$.

Proof. We prove that in the relative topology, there is a smooth structure on $f^{-1}(q)$ that makes $f^{-1}(q) \subset M$ a submanifold of dimension $\dim M - \dim N$. For this it is sufficient to prove that if $p \in f^{-1}(q)$, then there exists a coordinate system on a neighborhood U of $p \in M$ of which $f^{-1}(q) \cap U$ is a slice of that dimension and the transition maps are smooth.

Let $x_1, \dots, x_d$ be a coordinate system centered at $q \in N$. Since $d\varphi_p : T_pM \to T_qN$ is surjective, it follows from the corollary yesterday that $y_1 = x_1 \circ \varphi, \dots, y_d = x_d \circ \varphi$ form part of a coordinate system of $p \in M$. Complete the coordinate system to $y_1, \dots, y_d, y_{d+1}, \dots, y_l$. Then $\varphi^{-1}(q) \cap U$ is give by the slice

$$y_1 = 0, \dots, y_d = 0,$$

and we can take $y_{d+1}, \dots, y_l$ to be a coordinate system on $\varphi^{-1}(q) \cap U$. For smoothness of the transition maps, we leave the details to the readers. □

Theorem 8.2 (Constant Rank Theorem)**.** *Let $\varphi : M \to N$ be a smooth map such that the rank of $d\varphi_p$ is constant for all $p \in M$. Show that for each $p \in M$ and $\varphi(p) \in N$, there exists local coordinate systems (U, x) around p and (V, y) around $\varphi(p)$ such that*

$$y \circ \varphi \circ x^{-1}(x_1, \dots, x_k, x_{k+1}, \dots, x_d) = (x_1, \dots, x_k, 0, \dots, 0).$$

Example 8.1. *Consider the function $r : \mathbb{R}^{d+1} \to \mathbb{R}, r(x_1, \dots, x_{d+1}) = x_1^2 + \cdots + x_{d+1}^2$. Then we know*

$$(dr)_{(x_1,\dots,x_{d+1})} = 2x_1dx_1 + \cdots + 2x_{d+1}dx_{d+1}.$$

Therefore, we know that $dr \neq 0$ whenever $(x_1, \dots, x_{d+1}) \neq 0$. Hence $r^{-1}(c)$ is a d-dimensional submanifold. The tangent space at $(x_1, \dots, x_{d+1})$ is

$$\Big\langle (v_1, \dots, v_{d+1}) \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} v_i x_i = 0 \Big\rangle.$$

Example 8.2. *Consider the smooth map $F : M_{d\times d}(\mathbb{R}) \to M_{d\times d}(\mathbb{R}), F(A) = AA^T - I_d$. We show that $F^{-1}(0)$ is a smooth submanifold of dimension $d(d-1)/2$. This is usually called the orthogonal group $O(d)$.*

Write down all the components of the map $F((a_{ij})_{1\le i,j\le d}) = (\sum_{k=1}^d a_{ik}a_{jk} - \delta_{ij})_{1\le i,j\le d}$. First, note that the image of F lands in the space of all symmetric matrices $S_{d\times d}(\mathbb{R})$. Then we can compute the differential

$$(dF)_{(a_{ij})_{1\le i,j\le d}} = \Big(\sum_{k=1}^d a_{ik} da_{jk} + \sum_{k=1}^d a_{jk} da_{ik} \Big)_{1\le i,j\le d}.$$

Consider the differential at $A = I_d$. It is

$$(dF)_{(\delta_{ij})_{1\le i,j\le d}} = \Big(\sum_{k=1}^{d} \delta_{ik} da_{jk} + \sum_{k=1}^{d} \delta_{jk} da_{ik}\Big)_{1\le i,j\le d} = (da_{ji} + da_{ij})_{1\le i,j\le d}.$$

It follows that $(dF)_{I_d}$ is a surjection from $T_{I_d} M_{d\times d}(\mathbb{R}) = M_{d\times d}(\mathbb{R})$ to $T_O S_{d\times d}(\mathbb{R}) = S_{d\times d}(\mathbb{R})$, and the tangent space at I_d is the set of all skew symmetric matrices

$$\Big\langle (v_{ij})_{1\le i,j\le d} \mid (v_{ji} + v_{ij})_{1\le i,j\le d} = O \Big\rangle.$$

Finally, for any other $P \in F^{-1}(O)$, consider the smooth map

$$\varphi_P : M_{d\times d}(\mathbb{R}) \to M_{d\times d}(\mathbb{R}), \quad A \mapsto PA.$$

It has a smooth inverse $\varphi_{P^{-1}}$ and moreover $\varphi_P(F^{-1}(O)) = F^{-1}(O)$. This implies that $(dF)_P$ is also surjective at any $P \in F^{-1}(O)$. Therefore, O is a regular value of F and this finishes the proof.

9. Lecture 9: Tangent and Cotangent Bundles

Rather than describing the tangent and cotangent vectors at a single point on the manifold, we would often like to describe how the tangent and cotangent vectors smoothly change when the point moves around in the manifold. This requires us to put some topology on the space of all (co)tangent vectors.

Definition 9.1. *Let M be a smooth manifold. Then the tangent bundle TM and cotangent bundle T^*M of M are defined as*

$$TM = \bigcup_{p\in M} T_pM, \quad T^*M = \bigcup_{p\in M} T_p^*M.$$

The topology on TM is generated by the sets $\{(p, v) \in TM \mid p \in U, dx(v) \in \Omega\}$, where (U, x) is a coordinate system on M and Ω is an open subset in $\mathbb{R}^d$. The differentiable structure on TM is generated by the coordinate systems $(U \times \mathbb{R}^d, x \times dx)$, where (U, x) is a coordinate system on M and Ω is an open subset in $\mathbb{R}^d$.

It may seem like the tangent bundle is just the space $M \times \mathbb{R}^d$. However, this is rarely the case. In fact, the local coordinate maps secretly encode the twistings of the tangent spaces. For example, consider the coordinate systems on S^2 given by $S^2 \setminus \{n\}$ and $S^2 \setminus \{s\}$. Then on the overlap, one can see that the tangent spaces are identified in a nontrivial way. In fact, we can give the following alternative description:

Proposition 9.1. *Let M be a smooth manifold. Consider the topological space $\bigsqcup_{\alpha\in A} U_\alpha \times \mathbb{R}^d$. Define the equivalence relation such that*

$$(p, v_\alpha) \sim (p, v_\beta) \text{ if } p \in U_\alpha \cap U_\beta,\ dx_\alpha(v) = v_\alpha,\ dx_\beta(v) = v_\beta \text{ for some } v \in T_pM.$$

Then $\bigsqcup_{\alpha\in A} U_\alpha \times \mathbb{R}^d / \{(p, v_\alpha) \sim (p, v_\beta), v_\alpha = dx_\alpha \circ dx_\beta^{-1}(v_\beta)\} \cong TM$.

Example 9.1. *In the situation when $M = \mathbb{R}^d$, the proposition shows that we have*

$$T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d.$$

For a smooth map $\varphi : M \to N$, the push forward or the differential

$$d\varphi : TM \to TN, \quad (p, v_p) \mapsto (\varphi(p), d\varphi_p(v_p))$$

is a smooth map. Moreover, if $\varphi : M \to N$ is an embedding, we can show that $d\varphi : TM \to TN$ is also an embedding. However, the pull back map does NOT define a map between cotangent bundles. We will see how to resolve the issue on Friday.

Example 9.2. *Let $S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$. We have seen how to compute the tangent space of S^2. Namely, setting $r(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$, at $(x_1, x_2, x_3) \in S^2$, we have*

$$T_{(x_1,x_2,x_3)}S^2 = \{v \in \mathbb{R}^3 \mid dr(v) = 0\} = \{(v_1, v_2, v_3) \mid x_1v_1 + x_2v_2 + x_3v_3 = 0\}.$$

Then we know that $TS^2 \subset T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ is defined as the following subset:

$$TS^2 = \{(x_1, x_2, x_3; v_1, v_2, v_3) \mid x_1^2 + x_2^2 + x_3^2 = 1, x_1v_1 + x_2v_2 + x_3v_3 = 0\}.$$

We have seen that the tangent bundle and cotangent bundle are smooth manifolds. However, we also note that the transition maps $dx_\alpha \circ dx_\beta^{-1}$ are not only smooth maps, they are linear maps along the $\mathbb{R}^d$-directions. In the next lecture, we will explain the extra structure that can be put on a special type of manifolds.

10. Lecture 10: Vector Bundles

In this lecture, we introduce the notion of vector bundles, which are not only manifolds but have some linear vector space directions.

Definition 10.1. *Let M be a topological space. Then a rank-k vector bundle over M is a triple (π, E, M) where E is a topological space and $\pi : E \to M$ is a continuous map such that*

1. *for each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is a k-dimensional vector space;*
2. *for each $p \in M$, there is a neighbourhood U and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\pi_U \circ \Phi = \pi$, and $\Phi_q : \pi^{-1}(q) \to \mathbb{R}^k$ is a linear isomorphism.*

The pair (U, Φ) is called a vector bundle coordinate system or chart.

Definition 10.2. *Let M be a smooth manifold and (π, E, M) be a rank-k vector bundle. Then a C^k-differentiable vector bundle structure/atlas is a collection $\mathscr{F}$ of vector bundle charts (U_α, Φ_α) such that*

1. *$\bigcup_{\alpha \in A} U_\alpha = M$;*
2. *$\Phi_\alpha \circ \Phi_\beta^{-1}$ is a C^k-differentiable map for any $\alpha, \beta \in A$;*
3. *$\mathscr{F}$ is maximal so that, for any vector bundle chart (U, Φ), if $\Phi \circ \Phi_\alpha^{-1}$ and $\Phi_\alpha \circ \Phi^{-1}$ are C^k-differentiable for any $\alpha \in A$, then $(U, \Phi) \in \mathscr{F}$.*

In this case, (π, E, M) is called a smooth vector bundle of rank-k.

Example 10.1. *(1) The natural projection $\pi : M \times \mathbb{R}^k \to M$ defines a vector bundle on M. This is usually called the trivial vector bundle. (2) The first nontrivial example of a vector bundle is the Mobiüs bundle, which is a rank-1 vector bundle over the circle $\pi : E \to S^1$.*

Let us discuss the vector bundle structure more carefully. The condition that $\pi_{U_\alpha} \circ \Phi_\alpha = \pi = \pi_{U_\beta} \circ \Phi_\beta$ implies that the map $\Phi_\alpha \circ \Phi_\beta^{-1}$ can be written as the form

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)v),$$

where $g_{ij}(p) \in \mathrm{GL}(k, \mathbb{R})$ is a linear isomorphism. They are called the transition maps.

Lemma 10.1. *Let (π, E, M) be a rank-k (smooth) vector bundle on M and (U_α, Φ_α), (U_β, Φ_β) are local vector bundle coordinate charts. Then*

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)v),$$

and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ is a (smooth) map.

Using transition maps, we can reconstruct the vector bundle.

Lemma 10.2. *Let M be a manifold and $\{U_\alpha \mid \alpha \in A\}$ be an open cover. Let $\{g_{\alpha\beta} \mid \alpha, \beta \in A\}$ be a collection of smooth maps $U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that*

$$g_{\alpha\beta}(p)g_{\beta\gamma}(p) = g_{\alpha\gamma}(p), \quad \textit{for any } p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Then there exists a unique rank-k vector bundle (π, E, M) such that $\{g_{\alpha\beta} \mid \alpha, \beta \in A\}$ are the transition maps.

Proof. We can define the vector bundle by considering

$$E = \bigsqcup_{\alpha\in A} U_\alpha \times \mathbb{R}^k / \{(p, v_\alpha) \sim (p, v_\beta) \text{ if } p \in U_\alpha \cap U_\beta, v_\alpha = g_{\alpha\beta}(v_\beta)\}.$$

One can check that $\pi : E \to M, (p, v) \to p$ is a well defined (smooth) map and the collection $\mathrm{id} : U_\alpha \times \mathbb{R}^k \to U_\alpha \times \mathbb{R}^k$ defines a vector bundle structure. □

For vector bundles on a given topological space, there are many constructions that can be performed. It is an exercise to prove that these are vector bundles.

Definition 10.3. *Let (π_1, E_1, M) and (π_2, E_2, M) be vector bundles over M. Then a vector bundle map is a map $\varphi : E_1 \to E_2$ such that $\pi_1 = \pi_2 \circ \varphi$ and $\varphi_p : E_{1p} \to E_{2p}$ are linear maps between vector spaces. It is called a monomorphism if φ_p are injective, an epimorphism if φ_p are surjective, and an isomorphism if φ_p are isomorphic.*

Let (π, E, M) be a vector bundle. Then a subbundle is a vector bundle (π_0, E_0, M) where $E_0 \subseteq E$ and $\pi_0 = \pi|_{E_0}$. In other words suppose the transition maps of (π, E, M) is $\{g_{\alpha\beta} \mid \alpha, \beta \in A\}$, then the transition maps are

$$\{g_{0\alpha\beta} \mid \alpha, \beta \in A, i \circ g_{0\alpha\beta} = g_{\alpha\beta} \circ i\}.$$

Then the quotient bundle $(\overline{\pi}, E/E_0, M)$ is defined by fibers $(E/E_0)_p = E_p/E_{0p}$ and transition maps

$$\{\overline{g}_{\alpha\beta} \mid \alpha, \beta \in A\}$$

where $\overline{g}_{\alpha\beta} : \mathbb{R}^k/\mathbb{R}^l \to \mathbb{R}^k/\mathbb{R}^l$ is the quotient of $g_{\alpha\beta} : \mathbb{R}^k \to \mathbb{R}^k$.

Let (π_1, E_1, M) and (π_2, E_2, M) be vector bundles over M. Then the direct sum (or Whitney sum) is the vector bundle $(\pi_1 \oplus \pi_2, E_1 \oplus E_2, M)$ with fibers $(E_1 \oplus E_2)_p \cong E_{1p} \oplus E_{2p}$ and transition maps

$$\{g_{1\alpha\beta} \oplus g_{2\alpha\beta} \mid \alpha, \beta \in A\}.$$

The internal hom is the vector bundle $(\pi_{Hom}, Hom(E_1, E_2), M)$ with fibers $Hom(E_1, E_2)_p \cong Hom(E_{1p}, E_{2p})$ and transition maps

$$\{g_{2\alpha\beta}(-)g_{1\alpha\beta} \mid \alpha, \beta \in A\}.$$

In particular, the dual of (π, E, M) is the vector bundle (π^*, E^*, M) with fibers $E_p^* \cong (E_p)^*$ and transition maps $\{g_{\beta\alpha}^* \mid \alpha, \beta \in A\}$.

Using partitions of unity, many properties on vector spaces can be translated to vector bundles using the local-to-global argument.

Theorem 10.3. *Let M be a manifold and (π, E, M) is a vector bundle and (π_0, E_0, M) is a subbundle. Then there is an isomorphism of vector bundles $E \cong E_0 \oplus E/E_0$.*

Proof. Consider a vector space chart (U_α, Φ_α) of (π, E, M) and a vector bundle chart (U_α, Ψ_α) of (π_0, E_0, M). Then we have homeomorphisms $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ and $\Psi_\alpha : \pi_0^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^l$. Their composition have the form

$$\Phi_\alpha \circ \Psi_\alpha^{-1} : U_\alpha \times \mathbb{R}^l \to U_\alpha \times \mathbb{R}^k, \;\; (p, v_0) \mapsto (p, g_\alpha(v_0))$$

where $g_\alpha(p)$ is an injective linear map $\mathbb{R}^l \to \mathbb{R}^k$. Using implicit function theorem, we can define a projection map $h_\alpha(p) : \mathbb{R}^k \to \mathbb{R}^l$ such that $h_\alpha(p)g_\alpha(p) = \mathrm{id}$, which gives a projection map $f_\alpha(p) : E_p \to E_{0p}$. Then we use a partition of unity on M and define

$$f : E \to E_0, \quad f(p, v) = \Big(p, \sum\nolimits_{\alpha \in A} \varphi_\alpha(p) f_\alpha(p)(v)\Big).$$

This together with the natural quotient map defines the isomorphism $E \xrightarrow{\sim} E_0 \oplus E/E_0$. □

Remark 10.2. *The continuous version of the result also holds for all paracompact Hausdorff spaces.*

Given two topological spaces M and N and a map $\varphi : M \to N$, from a vector bundle on N we can naturally define a vector bundle on M:

Definition 10.4. *Let M and N be topological spaces or manifolds and $\varphi : M \to N$ a continuous or smooth map. Given a vector bundle (π, E, N) on N, the pull-back vector bundle on M is defined as $(\pi_\varphi, \varphi^* E, M)$ where*

$$\varphi^* E = \{(p, q, v) \mid \varphi(p) = \pi(q, v) = q\}, \quad \pi_\varphi(p, q, v) = p.$$

Using partition of unity, we can prove the following strong result, that a homotopy of vector bundle gives an isomorphism of vector bundle.

Theorem 10.4. *Let M be a manifold, I be an interval and $(\pi_I, E_I, M \times I)$ be a vector bundle on $M \times I$. Consider the vector bundle (π_t, E_t, M) where $E_t = \pi^{-1}(M \times \{t\})$ and $\pi_t = \pi|_{E_t}$. Then $(\pi_I, E_I, M \times I)$ is isomorphic to the pull-back bundle of (π_0, E_0, M). In particular, (π_0, E_1, M) is isomorphic to (π_1, E_1, M).*

Remark 10.3. *The continuous version of the result also holds for all paracompact Hausdorff spaces.*

Corollary 10.5. *Let M be a manifold, $f, g : M \to N$ be smoothly homotopic maps, and (π, E, N) a vector bundle over N. Then $(f^*\pi, f^*E, M)$ is isomorphic to $(g^*\pi, g^*E, N)$.*

Finally, we can resolve the issue that the pull back map of cotangent vectors do not define a map of cotangent bundles. Instead, the pull back map defines a vector bundle map

$$f^* : f^*T^*N \to T^*M, \quad (p, \varphi(p), \alpha_{\varphi(p)}) \mapsto (p, \varphi^* \alpha_p).$$

11. Lecture 11: Sections on Vector Bundles

The motivation we have at the beginning is to characterize what it means to have a smooth family of tangent vectors, and more generally, what it means to have a smooth family of vectors in a vector bundle.

Definition 11.1. *Let (π, E, M) be a (smooth) fiber bundle over M. A section is a (smooth) map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The zero section is the section $s : M \to E$ such that $s(p) = 0 \in E_p$ for any $p \in M$. The space of sections is denoted by $C^\infty(M, E)$.*

Lemma 11.1. *Let (π, E, M) be a (smooth) fiber bundle over M. Then $s : M \to E$ is a (smooth) section if and only if for any local coordinate system $(U_\alpha, x_\alpha, \Phi_\alpha)$,*

$$\Phi_\alpha \circ s(p) = (p, s_\alpha(p))$$

such that $s_\alpha : U_\alpha \to \mathbb{R}^k$ is a (smooth) map.

Proposition 11.2. *Let $\varphi : M \to N$ be a smooth map and (π, E, N) be a smooth vector bundle over N. Then there is a map between the space of smooth sections $C^\infty(N, E) \to C^\infty(M, \varphi^* E)$.*

Using partition of unity, we can construct sections on any vector bundles from a given vector at a point, just like we construct smooth functions.

Proposition 11.3. *Let (π, E, M) be a smooth fiber bundle over M. Then for any $v_p \in E_p$, there exists a section $s \in C^\infty(M, E)$ such that $s(p) = (p, v_p)$.*

12. Lecture 12: Vector Fields

In the last week, we started with the question of how to characterize a smooth family of (co)tangent vectors on a manifold, and ended by introducing vector bundles and sections, which assign each point on a manifold with a vector.

Now, we can define a vector field as a section in the tangent bundle.

Definition 12.1. *Let M be a smooth manifold. Then a smooth (resp. continuous) vector field X on a subset $U \subset M$ is a smooth (resp. continuous) section in the tangent bundle $X \in C^\infty(U, TM|_U)$ (resp. $X \in C^0(U, TM|_U)$).*

Proposition 12.1. *Let M be a smooth manifold and X be a vector field. Then X is smooth if and only if either of the conditions hold:*

(1) *for any local coordinate system (U, x),*
$$X|_U(p) = \sum_{i=1}^d a_i(p) \frac{\partial}{\partial x_i},$$
and $a_i(p) \in C^\infty(U)$;

(2) *for any open set $U \subset M$ and smooth function $f \in C^\infty(U)$, $Xf = df(X) \in C^\infty(U)$.*

For two vector fields, we can define the Lie bracket between them:

Definition 12.2. *Let M be a smooth manifold and X and Y be smooth vector fields. Then the Lie bracket $[X, Y]$ is a section in $(T^*M)^*$ defined by*
$$[X, Y]_p(f) = X_p(Yf) - Y_p(Xf).$$

Proposition 12.2. *Let M be a smooth manifold and X, Y and Z be smooth vector fields.*

(1) *$[X, Y]$ is a smooth vector field;*
(2) *(anti-symmetry) $[X, Y] = -[Y, X]$;*
(3) *(Jacobi identity) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$;*
(4) *for $f, g \in C^\infty(M)$, $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.*

Remark 12.1. *On a local coordinate system (U, x), suppose the vector fields are*
$$X|_U(p) = \sum_{i=1}^d a_i(p) \frac{\partial}{\partial x_i}, \quad Y|_U(p) = \sum_{i=1}^d b_i(p) \frac{\partial}{\partial x_i},$$
then their Lie bracket is given by the vector field
$$[X, Y]|_U(p) = \sum_{i,j=1}^d \left(a_i(p) \frac{\partial b_j}{\partial x_i}(p) - b_i(p) \frac{\partial a_j}{\partial x_i}(p) \right) \frac{\partial}{\partial x_j}.$$

Finally, we remark that push forwards of vector fields under smooth maps are not necessarily well defined, since a smooth map $M \to N$ may send different points on M to the same point on N.

13. Lecture 13: Intergral Curves and Flows

Given a vector field on a manifold, a point can move along the direction of the vector field and trace will form a curve. This is called the integral curve of the vector field.

Definition 13.1. *Let X be a smooth vector field on a manifold M. Then a smooth curve $\gamma : (a, b) \to M$ is called an integral curve of X if for any $t \in (a, b)$,*

$$\gamma'(t) = d\gamma\Big(\frac{d}{dt}\Big)_t = X(\gamma(t)).$$

Consider a local coordinate system (U, x), we can write down the differential equation for the integral curve of the vector field $X = \sum_{i=1}^d a_i(p)\partial/\partial x_i$:

$$\frac{d\gamma_i}{dt}(t) = a_i(\gamma(t)).$$

Since the coefficients $a_i(p)$ are smooth, the ordinary differential equation has a unique solution. In fact, let $U \subset \mathbb{R}^n$ be open subsets, $a : U \to \mathbb{R}^n$ be Lipschitz continuous. Then for any $p \in U$, there exists $\epsilon > 0$ such that

$$\frac{d\gamma_i}{dt}(t) = a_i(\gamma(t)), \quad \gamma(0) = p,$$

has a unique solution. This can be proved by considering the fixed point of the following mapping on Banach spaces $T : C^0((-\epsilon, \epsilon), \mathbb{R}^n) \to C^0((-\epsilon, \epsilon), \mathbb{R}^n)$:

$$(T\gamma)(t) = p + \int_0^t X(\gamma(s))ds.$$

This is a contraction mapping for sufficiently small $\epsilon > 0$ because

$$\|T\gamma - T\sigma\| = \sup_{t\in(-\epsilon,\epsilon)} \int_0^t |X(\gamma(s)) - X(\sigma(s))|ds \le C\epsilon\|\gamma - \sigma\|.$$

Then the contraction mapping principle ensures that the existence of a solution. The uniqueness of the solution follows from the computation that

$$|\gamma'(t) - \sigma'(t)| = |X(\gamma(t)) - X(\sigma(t))| \le C|\gamma(t) - \sigma(t)|, \quad |\gamma(t) - \sigma(t)| \le e^{Ct}|\gamma(0) - \sigma(0)|.$$

Therefore, we have the following global result on integral curves:

Theorem 13.1. *Let M be a smooth manifold and X be a smooth vector field on M. Then for any $p \in M$, there exists $-\infty \le a(p) < b(p) \le +\infty$ and a smooth maximal integral curve*

$$\gamma_p : (a(p), b(p)) \to M$$

of the vector field X such that for any integral curve $\sigma : (c, d) \to M$ with $\sigma(0) = p$ and $\gamma_p'(t) = X(\sigma(t))$, $(c, d) \subset (a(p), b(p))$ and $\sigma = \gamma|_{(c,d)}$.

Moreover, since the coefficients $a_i(p)$ are smooth, the unique solution to the ordinary differential equation depends smoothly on the initial condition $p \in M$ (the proof is much harder and we will omit that). Therefore, we have the following global result on integral flows:

Theorem 13.2. *Let M be a smooth manifold and X be a smooth vector field on M. For $t \in \mathbb{R}$, we can define a smooth map φ_t^X called the integral flow on the domain*

$$\mathscr{D}_t = \{p \in M \mid a(p) < t, b(p) > t\}$$

such that $\varphi_t^X(p) = \gamma_p(t)$ and

1. *for any $p \in M$, there is a neighbourhood U and $\epsilon > 0$ such that $U \subset \mathscr{D}_{-\epsilon} \cap \mathscr{D}_\epsilon$;*
2. *$\varphi_t^X : \mathscr{D}_t \to \mathscr{D}_{-t}$ and $\varphi_{-t}^X : \mathscr{D}_{-t} \to \mathscr{D}_t$ are inverse diffeomorphisms;*

(3) $\varphi_s^X \circ \varphi_t^X = \varphi_{s+t}^X$ *on the domain of* $\varphi_s^X \circ \varphi_t^X$.

Corollary 13.3. *Let* M *be a compact smooth manifold (without boundary) and* X *be a smooth vector field on* M*. For* $t \in \mathbb{R}$*, we can define a smooth diffeomorphism*

$$\varphi_t^X : M \to M, \quad \varphi_t^X(p) = \gamma_p(t)$$

such that $\varphi_s^X \circ \varphi_t^X = \varphi_{s+t}^X$.

Remark 13.1. *In general, a vector field* X *on* M *is complete if* $\mathscr{D}_t = M$ *for any* $t \in M$*. The corollary shows that all vector fields on compact manifolds (without boundary) are complete.*

As a corollary, one can show that a non-zero vector field always generate a local coordinate function.

Proposition 13.4. *Let* M *be a smooth manifold and* X *be a vector field such that* $X(p) \neq 0$*. Then there exists a local coordinate system* (U, x) *around* p *such that* $x(p) = 0$ *and*

$$X|_U = \frac{\partial}{\partial x_1}.$$

Proof. First, by corollaries of the inverse function theorem, we can find a local coordinate system (V, y) around p such that $X_p = (\partial/\partial y_1)_p$. Now consider the smooth map

$$\sigma(t, a_2, \ldots, a_d) = y \circ \varphi_t \circ y^{-1}(0, a_2, \ldots, a_d).$$

Then $d\sigma_0 \neq 0$ and by inverse function theorem, there is a small neighbourhood U of p such that σ is a diffeomorphism. Then $x = \sigma^{-1}$ satisfies the condition. $\square$

14. Lecture 14: Distributions and Frobenius Theorem

We have seen that a vector field generates integral curves. A natural generalization one can consider is to use vector subspaces to generate submanifolds.

Definition 14.1. *Let* M *be a* d*-dimensional smooth manifold. Then an* l*-dimensional distribution* $\mathscr{D}$ *is a choice of* l*-dimensional tangent subspaces* $\mathscr{D}_p \subset T_pM$ *for all* $p \in M$*.* $\mathscr{D}$ *is smooth if for each* $p \in M$ *there is a neighborhood* U *and* l *smooth vector fields* $X_1, \ldots, X_l$ *that span* $\mathscr{D}_q$ *for each* $q \in U$.

Definition 14.2. *We say that a vector field* X *belongs to (or lies in) the distribution* $\mathscr{D}$ *(denoted by* $X \in \mathscr{D}$*) if* $X_p \in \mathscr{D}_p$ *for all* $p \in M$*. A distribution* $\mathscr{D}$ *on* M *is incolutive if for any* $X, Y \in \mathscr{D}$*,* $[X, Y] \in \mathscr{D}$.

We say that a (injectively immersed) submanifold $\varphi : N \hookrightarrow M$ *is an integral manifold of a distribution* $\mathscr{D}$ *if* $d\varphi(T_qN) = \mathscr{D}_{\varphi(q)}$ *for all* $q \in N$.

Example 14.1. *Consider* $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ *and consider the vector field* $X = \partial/\partial x_1 + \alpha\partial/\partial x_2$ *for* $\alpha \notin \mathbb{Q}$*, which defines a 1-dimensional distribution. Then the curves* $\{(x_1, x_2) \mid x_1 \equiv \alpha x_2 + c\}$ *are integral manifolds of the distribution. They are only injectively immersed.*

The following proposition explains a necessary condition for a distribution to have an integral manifold and justifies the definition of involutive distributions.

Proposition 14.1. *Let* $\mathscr{D}$ *be a smooth distribution on* M *such that for each point of* M *there is an integral manifold* $\mathscr{D}$ *passing through. Then* $\mathscr{D}$ *is involutive.*

Proof. Consider an integral manifold $\varphi : N \hookrightarrow M$ of $\mathscr{D}$ with $\varphi(q) = p$. Consider two vector fields $X, Y \in \mathscr{D}$, Since we have an isomorphism $d\varphi : T_qN \xrightarrow{\sim} \mathscr{D}_{\varphi(q)}$, there exists X', Y' on N defined by

$$d\varphi \circ X' = X \circ \varphi, \quad d\varphi \circ Y' = Y \circ \varphi : N \to TM.$$

One can check that X', Y' are smooth vector fields and $d\varphi \circ [X', Y'] = [X, Y] \circ \varphi$, as

$$\begin{aligned}(d\varphi \circ [X', Y'])(f) &= [X', Y'](f \circ \varphi) = X'Y'(f \circ \varphi) - Y'X'(f \circ \varphi) \\ &= X'(d\varphi(Y')(f)) - Y'(d\varphi(X')(f)) = X'(Y(f) \circ \varphi) - Y'(X(f) \circ \varphi) \\ &= d\varphi(X')(Y(f)) - d\varphi(Y')(X(f)) = XY(f) - YX(f) = [X, Y](f),\end{aligned}$$

which completes the proof. □

Theorem 14.2 (Frobenius Theorem)**.** *Let $\mathscr{D}$ be an l-dimensional smooth involutive distribution on M. Then for any $p \in M$, there exists a coordinate system (U, x) around p such that the slices*

$$x_i = c_i, \quad l + 1 \leq i \leq d$$

are integral manifolds of $\mathscr{D}$, and if $\varphi : N \hookrightarrow U$ is a connected integral manifold, then $\varphi(N)$ lies in one of the slices.

Proof. We prove by induction on the dimension. Suppose $X_1, \ldots, X_l$ span $\mathscr{D}$ on a neighbourhood W in M. Consider a coordinate system (V, y) around p such that

$$X_1 = \partial/\partial y_1.$$

Then we try to consider the distribution $\mathscr{D}'$ spanned by

$$X_2' = X_2 - X_2(y_1)X_1, \ \ldots, \ X_l' = X_l - X_l(y_1)X_1.$$

We claim that this is still involutive. Note that $X_i'(y_1) = X_i(y_1) - X_i(y_1)X_1(y_1) = 0$. This means that $X_2', \ldots, X_l'$ lie in the slices $y_1 = c$. Since $\mathscr{D}$ is involutive, we can write

$$[X_i', X_j'] = \sum_{k=1}^{l} c_{ijk} X_k.$$

However, since $X_2', \ldots, X_l'$ lie in the slices $y_1 = c$, so is $[X_i', X_j']$. Therefore, we can write

$$[X_i', X_j'] = \sum_{k=2}^{l} c_{ijk} X_k = \sum_{k=2}^{l} c_{ijk} X_k'.$$

This shows that $\mathscr{D}'$ is still involutive. By the induction hypothesis, we can find a coordinate system (U, x) such that $x_1 = y_1$, and the slices

$$x_i = c_i, \quad l + 1 \leq i \leq d$$

are integral manifolds of $\mathscr{D}'$. Finally, we notice that the equality $X_i'(x_j) = 0$ for $l+1 \leq j \leq d$ will also imply that $X_i(x_j) = 0$ for $l + 1 \leq j \leq d$. This will complete the proof. □

Moreover, similar to the situation of integral curves, we can find a maximal integral submanifold for an involutive distribution.

Definition 14.3. *A maximal integral manifold $\varphi : N \hookrightarrow M$ of a distribution $\mathscr{D}$ on a manifold M is a connected integral manifold of $\mathscr{D}$ such that for any connected integral manifold $\varphi' : N' \hookrightarrow M$ of $\mathscr{D}$, if $\varphi(N) \subset \varphi'(N')$, then $\varphi(N) = \varphi'(N')$.*

Theorem 14.3. *Let $\mathscr{D}$ be a smooth involutive distribution on M. Then for any $p \in M$ there exists a unique maximal connected integral manifold of $\mathscr{D}$ passing through, and every connected integral manifold of $\mathscr{D}$ through p is contained in the maximal one.*

15. Lecture 15: Tensor Algebra

We recall some basic notions in multilinear algebra. First, we recall the tensor product.

Definition 15.1. *Let V and W be vector spaces over a field. Consider the free vector space generated by elements $(v, w) \in V \times W$*

$$F(V, W) = \Big\{ \sum_{i=1}^{k} \lambda_i (v_i, w_i) \mid (v_i, w_i) \in V \times W \Big\}.$$

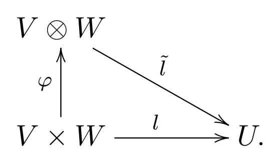
Let $R(V, W) \subset F(V, W)$ be the subspace generated by the collections

$$\begin{aligned} &(v_1 + v_2, w) - (v_1, w) - (v_2, w), \\ &(v, w_1 + w_2) - (v, w_1) - (v, w_2), \\ &(av, w) - a(v, w), \quad (v, aw) - a(v, w). \end{aligned}$$

Then we define the tensor product to be $V \otimes W = F(V, W)/R(V, W)$, and denote the equivalence class of (v, w) by $v \otimes w$.

Here are some basic properties of the tensor product. We will omit the proofs.

Proposition 15.1 (Universal property)**.** *Let V and W be vector spaces over a field. Let φ denote the bilinear map $V \times W \to V \otimes W, (v, w) \mapsto v \otimes w$. Then whenever U is a vector space and $l : V \times W \to U$ is a bilinear map, there exists a unique linear map $\tilde{l} : V \otimes W \to U$ such that the following diagram commutes:*



Proposition 15.2. *Let V and W be vector spaces over a field.*

1. *$V \otimes W$ is naturally isomorphic to $W \otimes V$;*
2. *$(V \otimes W) \otimes U$ is naturally isomorphic to $V \otimes (W \otimes U)$;*
3. *$\dim V \otimes W = (\dim V)(\dim W)$, and a basis $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$ for V and W determines a basis $\{v_1 \otimes w_1, \dots, v_m \otimes w_n\}$ of $V \otimes W$.*

Then we define the tensor algebra, consisting of tensors of arbitrary lengths.

Definition 15.2. *Let V be a vector space over a field. Then the tensor space of type (r, s) is defined by $V_{r,s} = V^{\otimes r} \otimes (V^*)^{\otimes s}$. The two-sided tensor algebra of V is the noncommutative algebra*

$$T_{\pm}(V) = \bigoplus_{r,s \geq 0} V_{r,s},$$

where the product is given by the tensor product $(v, w) \mapsto v \otimes w$. We define the (one-sided) tensor algebra to be the noncommutative algebra

$$T(V) = \bigoplus_{r \geq 0} V_{r,0}.$$

We write $T^k(V) = V_{k,0} = V^{\otimes k}$.

Consider the natural pairing between a vector space V and its dual space V^* together with the universal property, we also get a natural pairing between $T^*(V)$ and $T^*(V^*)$ as follows:

Lemma 15.3. *Let V be a vector space over a field. Then there exists a natural non-degenerate pairing between $V_{r,s}$ and $(V^*)_{s,r}$ given by*

$$(v_1 \otimes \cdots \otimes v_r \otimes v_1^* \otimes \cdots \otimes v_s^*, u_1 \otimes \cdots \otimes u_s \otimes u_1^* \otimes \cdots \otimes_r^*),\ u_1^*(v_1)\dots u_r^*(v_r)\cdot v_1^*(u_1)\dots v_s^*(u_s).$$

In particular, when $\dim V < \infty$, $(V_{r,s})^* \simeq (V^*)_{s,r}$.

We can define the tensor bundles of a manifold and tensor products of vector bundles over a manifold as follows:

Definition 15.3. *Let (π_1, E_1, M) and (π_2, E_2, M) be vector bundles over a space M. Then the tensor product is the vector bundle $(\pi_1 \otimes \pi_2, E_1 \otimes E_2, M)$ with fibers $(E_1 \otimes E_2)_p \cong E_{1p} \otimes E_{2p}$ and transition maps*

$$\{g_{1\alpha\beta} \otimes g_{2\alpha\beta} \mid \alpha, \beta \in A\}.$$

Definition 15.4. *Let M be a smooth manifold. Then the tensor bundle $T_{r,s}M$ of M are defined as*

$$T_{r,s}M = \bigcup_{p\in M} (T_pM)_{r,s}.$$

The topology on $T_{r,s}M$ is generated by the sets $\{(p, v_1 \otimes v_r \otimes v_1^* \otimes \dots, v_s^*) \in TM \mid p \in U, x_*(v_i) \in \Omega_i, (x^{-1})^*(v_j^*) \in \Omega_j^*\}$, where (U, x) is a coordinate system on M and Ω_i and Ω_j^* are open subsets in $\mathbb{R}^d$. The differentiable structure on TM is generated by the coordinate systems $(U \times \mathbb{R}^d, x \times x_* \otimes \cdots \otimes x_* \otimes (x^{-1})^* \otimes \cdots \otimes (x^{-1})^*)$, where (U, x) is a coordinate system on M and Ω is an open subset in $\mathbb{R}^d$. A tensor field on M of type (r, s) is a section of the tensor bundle $T_{r,s}M$.

Remark 15.1. *Under a coordinate system (U, x) on M, a tensor field of type (r, s) is of the form*

$$\alpha(p) = \sum_{i_1,\dots,i_r,j_1,\dots,j_s} a_{i_1\dots i_r j_1\dots j_s}(x(p)) \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_1}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}.$$

16. Lecture 16: Exterior Algebra

In the tensor algebra of a vector space, we are particularly interested in symmetric algebras and exterior algebras. We introduce their definitions now.

Definition 16.1. *Consider the tensor algebra of a vector space $T(V)$. Define the subspace $J(V) \subset T(V)$ be the vector space generated by*

$$v_1 \otimes \cdots \otimes (v_i \otimes v_{i+1} - v_{i+1} \otimes v_i) \otimes \cdots \otimes v_r.$$

Then the symmetric algebra is $S(V) = Sym(V) = T(V)/J(V)$, where the equivalence class of $v_1 \otimes \cdots \otimes v_r$ is denoted by $v_1 \cdots v_r$, and the product is induced by the tensor product. We write $S^k(V) = Sym^k(V) = T^k(V)/J^k(V)$ where $J^k(V) = J(V) \cap T^k(V)$.

Define the subspace $I(V) \subset T(V)$ be the vector space generated by

$$v_1 \otimes \cdots \otimes (v_i \otimes v_i) \otimes \cdots \otimes v_r.$$

Then the exterior algebra is $\Lambda(V) = T(V)/I(V)$, where the equivalence class of $v_1 \otimes \cdots \otimes v_r$ is denoted by $v_1 \wedge \cdots \wedge v_r$, and the product is induced by the tensor product. We write $\Lambda^k(V) = T^k(V)/I^k(V)$ where $I^k(V) = I(V) \cap T^k(V)$.

Definition 16.2. *A multilinear map $h : V \otimes \cdots \otimes V \to \mathbb{R}$ is called symmetric if for any $\sigma \in S_n$,*

$$h(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = h(v_1, \dots, v_n).$$

It is called alternating if for any $\sigma \in S_n$,

$$h(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \operatorname{sgn}(\sigma) h(v_1, \dots, v_n).$$

The symmetric algebra and the exterior algebra also satisfies the universal properties:

Proposition 16.1 (Universal Property)**.** *Let V be a vector space over a field.*

(1) *Let φ_s denote the bilinear map $V \times \cdots \times V \to S^k(V), (v_1, v_2, \ldots, v_k) \mapsto v_1 v_2 \ldots v_k$. Then whenever U is a vector space and $l : V \times \cdots \times V \to U$ is a symmetric multilinear map, there exists a unique linear map $\tilde{l} : S^k(V) \to U$ such that the following diagram commutes:*

$$\begin{array}{ccc} S^k(V) & & \\ \uparrow{\scriptstyle \varphi_s} & \searrow^{\tilde{l}} & \\ V \times \cdots \times V & \xrightarrow{\;l\;} & U. \end{array}$$

(2) *Let φ_a denote the bilinear map $V \times \cdots \times V \to \Lambda^k(V), (v_1, v_2, \ldots, v_k) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_k$. Then whenever U is a vector space and $l : V \times \cdots \times V \to U$ is an alternating multilinear map, there exists a unique linear map $\tilde{l} : \Lambda^k(V) \to U$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Lambda^k(V) & & \\ \uparrow{\scriptstyle \varphi_a} & \searrow^{\tilde{l}} & \\ V \times \cdots \times V & \xrightarrow{\;l\;} & U. \end{array}$$

Proposition 16.2. *Let V be a vector space over a field.*

(1) *If $u \in \Lambda_k(V)$ and $v \in \Lambda_l(V)$, then $u \wedge v = (-1)^{kl} v \wedge u$;*
(2) *if $\dim V = n$, then $\dim \Lambda^k(V) = \binom{n}{k}$, and a basis $\{v_1, \ldots, v_n\}$ of V determines a basis $\{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$ of $\Lambda^k(V)$;*
(3) *if $\dim V = n$, then $\dim S^k(V) = \sum_{i=0}^{k} \binom{n}{i}\binom{i+k}{i}$, and a basis $\{v_1, \ldots, v_n\}$ of V determines a basis $\{v_{i_1} v_{i_2} \ldots v_{i_k} \mid i_1 \leq i_2 \leq \cdots \leq i_k\}$ of $S^k(V) = Sym^k(V)$;*

Using the natural pairing between a vector space V and its dual space V^* together with the universal property, we also get a natural pairing between $\Lambda^*(V)$ and $\Lambda^*(V^*)$ as follows:

Lemma 16.3. *Let V be a vector space over a field. Then there exists a natural non-degenerate pairing between $\Lambda^k(V)$ and $\Lambda^k(V^*)$ given by*

$$(v_1 \wedge \cdots \wedge v_k, u_1^* \wedge \cdots \wedge u_k^*), \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) u_1^*(v_{\sigma_1}) \ldots u_k^*(v_{\sigma_k}) = \det(u_i^*(v_j)).$$

In particular, when $\dim V < \infty$, $\Lambda^k(V)^* \simeq \Lambda^k(V^*)$.

Definition 16.3. *Let V be a vector space over a field. Then for any $v \in V$, the interior multiplication by v is defined by*

$$i(v) : \Lambda^*(V^*) \to \Lambda^*(V^*), \quad (i(v)u^*, w) = (u^*, v \wedge w).$$

Lemma 16.4. *Let V be a vector space over a field. Then $i(v) : \Lambda^*(V^*) \to \Lambda^*(V^*)$ is an antiderivation, that is,*

$$i(v)(u_1^* \wedge u_2^*) = i(v)u_1^* \wedge u_2^* + (-1)^k u_1^* \wedge i(v)u_2^*, \quad u_1^* \in \Lambda^k(V^*), u_2^* \in \Lambda^l(V^*).$$

We can define the symmetric and exterior tensor bundles of a manifold and symmetric and exterior products of vector bundles over a manifold as follows:

Definition 16.4. *Let (π, E, M) be a vector bundle over a space M. Then the symmetric tensor product is the vector bundle $(S^k(\pi), S^k(E), M)$ with fibers $(S^k(E))_p \cong S^k(E_p)$ and transition maps*

$$\{g_{\alpha\beta}^k \mid \alpha, \beta \in A\}.$$

the exterior tensor product is the vector bundle $(\Lambda^k(\pi), \Lambda^k(E), M)$ *with fibers* $(\Lambda^k(E))_p \cong \Lambda^k(E_p)$ *and transition maps*

$$\{g^{\wedge k}_{\alpha\beta} \mid \alpha, \beta \in A\}.$$

Definition 16.5. *Let M be a smooth manifold. Then the symmetric and exterior tensor bundle $S^k(T^*M)$ and $\Lambda^k(T^*M)$ of M are defined as*

$$S^k(T^*M) = \bigcup_{p\in M} S^k(T_p^*M), \quad \Lambda^k(T^*M) = \bigcup_{p\in M} \Lambda^k(T_p^*M).$$

The topology is generated by local coordinate systems on M.

Example 16.1. *(1) Consider the symmetric tensor bundle $S^2(T^*M)$. Then under a local coordinate system, a section is of the form $g(p) = \sum_{i,j=1}^d g_{ij}(x(p))dx_i dx_j$. Suppose $g(p) : T_pM \otimes T_pM \to \mathbb{R}$ is positive definite. Then g is called a Riemannian metric on M.*

(2) Consider the exterior tensor bundle $\Lambda^k(T^*M)$. Then under a local coordinate system, a section is of the form $\omega(p) = \sum_{i_1<\cdots<i_k} \omega_{i_1\ldots i_k}(x(p))dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. This is called a differential form. When $k = d$, under a local coordinate system, a section is of the form $\omega(p) = \omega(x(p))dx_1 \wedge \cdots \wedge dx_d$. Suppose $\omega(p) \neq 0$. Then ω is called a volume form.

17. Lecture 17: Differential Forms

In the last lecture, we defined the exterior tensor product of vector bundles. In this lecture, we will focus on the exterior tensor product of the cotangent bundles. Sections of these vector bundles are called differential forms.

Definition 17.1. *Let M be a smooth manifold. Then the space of differential k-forms $\Omega^k(M)$ is the space of sections on the k-th exterior tensor product of the cotangent bundle:*

$$\Omega^k(M) = \Gamma(M, \Lambda^k(T^*M)).$$

Remark 17.1. *Consider a coordinate system (U, x) in M, then a differential form ω can be written as*

$$\omega(p) = \sum_{i_1<\cdots<i_k} \omega_{i_1\ldots i_k}(x(p))dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Given two differential forms ω and ω', we can take their sum $\omega + \omega'$, their exterior product $\omega \wedge \omega'$, and given a function f, we can take the multiplication of a form by a function $f\omega$.

Given a differential k-form ω and k vector fields $X_1, \ldots, X_k$, using the pairing of exterior tensor bundles

$$\Lambda^k(T^*M) \times \Lambda^k(TM) \to M \times \mathbb{R},$$

we can define the pairing which gives a smooth function

$$\omega(X_1, \ldots, X_k)(p) = \omega_p(X_1(p), \ldots, X_k(p)).$$

Conversely, we can show that any alternating $C^\infty(M)$-linear map on the space of vector fields $\mathfrak{X}(M)$ can be defined this way:

Lemma 17.1. *For any alternating $C^\infty(M)$-linear map $l : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$, there is a canonical k-form $\omega_l \in \Omega^k(M)$ so that $\omega_l(X_1, \ldots, X_k) = l(X_1, \ldots, X_k)$.*

Proof. First, we claim that the value $l(X_1, \ldots, X_k)(p)$ only depends on $X_1(p), \ldots, X_k(p)$. For simplicity, we only consider $C^\infty(M)$-linear maps $l : \mathfrak{X}(M) \to C^\infty(M)$ and show that $l(X)(p)$ only depends on $X(p)$. Using linearity, it suffices to show that $l(X)(p) = 0$ as long as $X(p) = 0$. Consider a local coordinate system (U, x) and write $X|_U = \sum_{i=1}^d a_i(\partial/\partial x_i)$

where $a_i(p) = 0$. Consider a compact neighbourhood $V \subset U$ of p and a bump function φ on $V \subset U$. Then setting the globally defined vector fields $X_i = \varphi a_i(\partial/\partial x_i)$, we can write

$$X = \sum_{i=1}^{d} X_i + (1-\varphi)X.$$

Then, by the $C^\infty(M)$-linearity, we have

$$l(X)(p) = \sum_{i=1}^{d} l(X_i)(p) + (1-\varphi(p))l(X)(p) = 0.$$

Now, since $l(X_1, \ldots, X_k)(p)$ only depends on $X_1(p), \ldots, X_k(p)$, we can define the value of k-form ω_l at $p \in M$ by setting $(\omega_l)_p(v_1, \ldots, v_k) = l(X_1, \ldots, X_k)$ where $X_1, \ldots, X_k$ are any smooth vector fields that satisfy $X_1(p) = v_1, \ldots, X_k(p) = v_k$. This completes the proof. □

For smooth functions, we have defined the notion of the differential, which is a linear map $d : \Omega^0(M) \to \Omega^1(M)$. Now, we will extend the differential to a linear map on all differential forms: $d : \Omega^*(M) \to \Omega^{*+1}(M)$:

Theorem 17.2. *There exists a unique antiderivation $d : \Omega^*(M) \to \Omega^{*+1}(M)$ such that*
(1) $d^2 = 0$*;*
(2) $d : \Omega^0(M) \to \Omega^1(M)$ *is the differential map;*
(3) $d(a_1\omega_1 + a_2\omega_2)|_p = a_1 d\omega_1|_p + a_2 d\omega_2|_p$ *if* $\omega_1, \omega_2 \in \Omega^k(M)$*;*
(4) $d(\omega_1 \wedge \omega_2)|_p = d\omega_1|_p \wedge \omega_2|_p + (-1)^k \omega_1|_p \wedge d\omega_2|_p$ *if* $\omega_1 \in \Omega^k(M), \omega_2 \in \Omega^l(M)$.

Moreover, $d\omega_p = d\omega'_p$ if $\omega|_U = \omega'|_U$ on some neighbourhood U.

Proof. First, we show the uniqueness. Let d be any exterior differential satisfying properties (1)–(4). We prove that $d\omega_p = 0$ if $\omega|_U = 0$ on some neighbourhood U. In fact, let φ be a bump function on some neighbourhood $U' \subset U$ of p. Then $\omega = (1-\varphi)\omega$. We can compute using (4) that

$$d\omega_p = d((1-\varphi)\omega)_p = d(1-\varphi)_p \wedge \omega_p + (1-\varphi(p))d\omega_p = 0.$$

Then, we can define the exterior differential d on local coordinate systems (U, x). Given $\omega = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, by property (1)–(4), $d\omega$ must be of the form

$$d\omega = \sum_{i_1 < \cdots < i_k} da_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Using properties (1)–(4), we show that for any exterior differential d' defined using a different cooridinate system, we have

$$\begin{aligned}
d'\omega &= d'(a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \\
&= d'a_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} + \sum\nolimits_{j=1}^{k} a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge d'dx_{i_j} \wedge \cdots \wedge dx_{i_k} \\
&= da_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} + \sum\nolimits_{j=1}^{k} a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dd^2 x_{i_j} \wedge \cdots \wedge dx_{i_k} \\
&= da_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = d\omega.
\end{aligned}$$

Then, we show the existence. Consider a coordinate system (U, x), and define the exterior differential of ω by

$$d\omega = \sum_{i_1 < \cdots < i_k} da_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Then we show that the $d\omega$ we defined satisfies conditions (1)–(4). By the formula we write down, (2)–(3) are obvious. (4) follows from the Leibniz rule:

$$\begin{aligned} d((&a_{i_1\dots i_k}dx_{i_1}\wedge\cdots\wedge dx_{i_k})\wedge(b_{j_1\dots j_l}dx_{j_1}\wedge\cdots\wedge dx_{j_l})) \\ &= d(a_{i_1\dots i_k}b_{j_1\dots j_l}dx_{i_1}\wedge\cdots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_l}) \\ &= (a_{i_1\dots i_k}db_{j_1\dots j_l}+b_{j_1\dots j_l}da_{i_1\dots i_k})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_l}, \\ d(&a_{i_1\dots i_k}dx_{i_1}\wedge\cdots\wedge dx_{i_k})\wedge(b_{j_1\dots j_l}dx_{j_1}\wedge\cdots\wedge dx_{j_l}) \\ &\quad+(-1)^k(a_{i_1\dots i_k}dx_{i_1}\wedge\cdots\wedge dx_{i_k})\wedge d(b_{j_1\dots j_l}dx_{j_1}\wedge\cdots\wedge dx_{j_l}) \\ &= b_{j_1\dots j_l}da_{i_1\dots i_k}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_l} \\ &\quad+a_{i_1\dots i_k}db_{j_1\dots j_l})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_l}. \end{aligned}$$

Finally, (1) follows from the property that $d^2f=0$:

$$d^2f=d(df)=d\Big(\sum_{i=1}^d\frac{\partial f}{\partial x_i}dx_i\Big)=\sum_{j=1}^d\sum_{i=1}^d\frac{\partial^2 f}{\partial x_i\partial x_j}dx_j\wedge dx_i=0.$$

The fact that the definition is independent of coordinate changes follows from the uniqueness. □

Just as vectors can pair with covectors, vector fields can pair with differential forms.

Definition 17.2. *Let X be a smooth vector field on M, and let $\omega\in\Omega^k(M)$. The interior multiplication of ω by X is the form $i(X)\omega$ whose value at $p\in M$ is the interior multiple $i(X_p)\omega_p$.*

Finally, we show that differential forms are functorial with respect to smooth maps.

Definition 17.3. *Let $\varphi: M\to N$ be a smooth map. Then define $\varphi^*:\Lambda^*T^*_{\varphi(p)}N\to\Lambda^*T^*_pM$ to be the natural map induced by $\varphi^*: T^*_{\varphi(p)}N\to T^*_pM$. Let $\omega\in\Omega^k(M)$. The pull-back of ω by φ is a form $\varphi^*\omega$ whose value at $p\in M$ is the pull-back $\varphi^*\omega_p$.*

Proposition 17.3. *Let $\varphi: M\to N$ be a smooth map. Then*
1. *$\varphi^*:\Omega^*(N)\to\Omega^*(M)$ is a ring/algebra homomorphism;*
2. *$\varphi^*d\omega=d(\varphi^*\omega)$ for any $\omega\in\Omega^*(N)$;*
3. *$(\varphi^*\omega)_p(X_{1p},\dots,X_{kp})=\omega_{\varphi(p)}(\varphi_*X_{1p},\dots,\varphi_*X_{kp})$.*

18. Lecture 18: Lie Derivatives

Since the (co)tangent spaces at different points are not canonically identified, it is harder to define derivatives of vector fields or differentials forms. Any such definition will require a choice of the identification of (co)tangent spaces of nearby points. We now define the derivatives of tensor fields and differential forms along a vector field called the Lie derivative, where we identity nearby (co)tangent spaces using the integration flow. (Note that this is different from the so-called covariant derivatives in Riemannian geometry.)

Definition 18.1. *Let X and Y be vector fields on a manifold M and $\varphi^t_X: U\to M$ be integration flow of X in a neighbourhood U of p. Then the Lie derivative of the vector field Y along X is*

$$(L_XY)_p=\frac{d}{dt}\left((\varphi_X^{-t})_*Y_{\varphi^t_X(p)}\right)|_{t=0}.$$

Let ω be a differential form on the manifold M. Then the Lie derivative of the differential form ω along X is

$$(L_X\omega)_p=\frac{d}{dt}\left((\varphi_X^{t})^*\omega_{\varphi^t_X(p)}\right)|_{t=0}.$$

Roughly, the idea in the definition is to use the integration flow φ_X^t to identity the (co)tangent spaces at p and $\varphi_X^t(p)$, and then compare the (co)tangent vectors at these points.

The following proposition shows that this is a reasonable definition, since it agrees with the usual derivative on smooth functions:

Proposition 18.1. *Let X be a vector field on M.*

1. *Let f be a function on M. Then $L_X f = X(f)$;*
2. *Let Y be a vector field on M. Then $L_X Y = [X, Y]$;*
3. *Let ω be a differential form on M. Then $L_X\omega = di(X)\omega + i(X)d\omega$;*
4. *Let ω be a differential form and $X_1, \ldots, X_k$ be vector fields on M. Then*

$$L_X(\omega(X_1, \ldots, X_k)) = (L_X\omega)(X_1, \ldots, X_k) + \sum_{i=1}^{k} \omega(X_1, \ldots, L_X X_i, \ldots, X_k).$$

Remark 18.1. *(3) is also called Cartan's magic formula.*

Proof. First, we show that $L_X(Y(f)) = (L_X Y)(f) + Y(L_X f)$. This is because

$$\begin{aligned} L_X(Y(f)) &= \lim_{t\to 0} \frac{(\varphi_X^t)^*(Y(f)) - Y(f)}{t} \\ &= \lim_{t\to 0} \frac{(\varphi_X^{-t})_* Y((\varphi_X^t)^* f) - (\varphi_X^{-t})_* Y(f)}{t} + \lim_{t\to 0} \frac{((\varphi_X^{-t})_* Y)(f) - Y(f)}{t} \\ &= \lim_{t\to 0} \frac{Y((\varphi_X^t)^* f) - Y(f)}{t} + \lim_{t\to 0} \frac{((\varphi_X^{-t})_* Y)(f) - Y(f)}{t} \\ &= Y(L_X f) + (L_X Y)(f). \end{aligned}$$

We can conclude that (4) holds in a similar way. Then, using (4), we can conclude that (2) holds:

$$(L_X Y)(f) = L_X(Y(f)) - Y(L_X f) = X(Y(f)) - Y(X(f)) = [X, Y](f).$$

Finally, we can show that (3) holds by the following local computation:

$$\begin{aligned} L_X\Big(\sum_{i=1}^d a_i dx_i\Big)(Y) &= \sum_{i=1}^d (L_X a_i) dx_i(Y) + a_i (L_X dx_i)(Y) \\ &= \sum_{i=1}^d (Xa_i)(Yx_i) + a_i X(Y(x_i)) - a_i[X, Y](x_i) \\ &= \sum_{i=1}^d (Xa_i)(Yx_i) - (Ya_i)(Xx_i) + Y(a_i X(x_i)) \\ &= \sum_{i=1}^d da_i(X) dx_i(Y) - dx_i(X) da_i(Y) + d(a_i dx_i(X))(Y). \end{aligned}$$

Then we can conclude the proof by induction on the degree. □

Finally, using the above proposition, we can give the following coordinate free formula for computing the exterior differential:

Proposition 18.2. *Let ω be a differential form and $X_1, X_2, \ldots, X_{k+1}$ be vector fields on M. Then*

$$\begin{aligned} d\omega(X_1, X_2, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i \omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}). \end{aligned}$$

Proof. We prove by induction on the degree using Cartan's formula. For a 1-form ω, we can compute

$$\begin{aligned} d\omega(X,Y) &= (i(X)d\omega)(Y) = (L_X\omega)(Y) - (di(X)\omega)(Y) \\ &= L_X(\omega(Y)) - \omega(L_XY) - d(\omega(X))(Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]). \end{aligned}$$

Then we complete the proof by induction. □

Remark 18.2. *(1) Under some local coordinate systems (U,x), let $X = \sum_{i=1}^d a_i\partial/\partial x_i$ and $Y = \sum_{i=1}^d b_i\partial/\partial x_i$, the Lie derivative on vector fields can be computed by*

$$L_XY = [X,Y] = \sum_{i,j=1}^d \Big(a_j\frac{\partial b_i}{\partial x_j} - b_j\frac{\partial a_i}{\partial x_j}\Big)\frac{\partial}{\partial x_i}.$$

(2) Under some local coordinate systems (U,x), let $X = \sum_{i=1}^d a_i\partial/\partial x_i$ and $\omega = \sum_{i=1}^d b_i dx_i$, the Lie derivative on 1-forms can be computed by

$$L_X\omega = \sum_{i,j=1}^d \Big(a_j\frac{\partial b_i}{\partial x_j} + b_j\frac{\partial a_i}{\partial x_j}\Big)dx_i.$$

19. Lecture 19: Differential Ideals

In the previous lecture, we have seen the relationship between the exterior differential and the Lie bracket. In this lecture, we will review the Frobenius theorem for distributions, which was stated in terms of Lie brackets of vector fields, and reformulate it in terms of differntials of forms.

This was one of the original motivations that E. Cartan introduced the calculus of differential forms, and we will see why the antisymmetry of differntial forms play a crucial role in the problem.

Definition 19.1. *Let $\mathscr{D}$ be an l-dimensional smooth distribution on M. A k-form ω is said to annihilate $\mathscr{D}$ if for each $p \in M$ and any $v_1, \dots, v_k \in \mathscr{D}(p)$,*

$$\omega_p(v_1, \dots, v_k) = 0.$$

Define $\mathscr{I}(\mathscr{D}) = \{\omega_0 + \cdots + \omega_d \mid \omega_i \in \Omega^i(M) \text{ annihilates } \mathscr{D}\}$.

Definition 19.2. *A collection $\omega_1, \dots, \omega_l$ of 1-forms on M is called independent if they form an independent set in T_p^*M for each $p \in M$.*

We can show that the annihilator $\mathscr{I}(\mathscr{D})$ is an ideal of the ring $\Omega^*(M)$ and every distribution corresponds to such an ideal and vice versa.

Proposition 19.1. *Let $\mathscr{D}$ be an l-dimensional smooth distribution on M.*

(1) *$\mathscr{I}(\mathscr{D})$ is an ideal of $\Omega^*(M)$: for any $\omega \in \mathscr{I}(\mathscr{D})$ and $\eta \in \Omega^*(M)$, $\omega \wedge \eta \in \mathscr{I}(\mathscr{D})$;*
(2) *$\mathscr{I}(\mathscr{D})$ is locally generated by $d-l$ independent 1-forms: $\omega \in \mathscr{I}(\mathscr{D})$ iff for any $p \in M$, there is a neighbourhood U and 1-forms $\omega_1, \dots, \omega_{d-l}$ on U such that $\omega|_U$ is generated by these 1-forms.*

Conversely, let $\mathscr{I}$ be an ideal of $\Omega^*(M)$ locally generated by $d-l$ independent 1-forms, there exists an l-dimensional smooth distribution $\mathscr{D}$ on M such that $\mathscr{I} = \mathscr{I}(\mathscr{D})$.

Proof. We only prove statement (2). Suppose $\mathscr{D}$ is locally generated by vector fields $X_1, \dots, X_l$ on an open subset U. Then by inverse function theorem, there exist vector fields $X_{l+1}, \dots, X_d$ such that $X_1, \dots, X_l, X_{l+1}, \dots, X_d$ form a basis of tangent spaces at all points in U. We take the dual 1-forms $\omega_1, \dots, \omega_d$ such that

$$\omega_i(X_j)|_U = \delta_{ij}, \quad 1 \le i,j \le d.$$

For $\omega \in \Omega^*(M)$, we write $\omega|_U = \sum_{i_1<\cdots<i_k} a_{i_1\ldots i_k}\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$. Then if $\omega \in \mathscr{I}(\mathscr{D})$, it must be the case that $a_{i_1\ldots i_k} = 0$ whenever there exists $i_j \in \{1,\ldots,l\}$. Conversely, if $\omega|_U$ is generated by $\omega_1,\ldots,\omega_{d-l}$, then clearly $\omega \in \mathscr{I}(\mathscr{D})$. This proves statement (2). □

In the previous weeks, we have seen that a distribution is integrable if and only if it is involutive, namely, it is closed under Lie bracket. Using the exterior differential, we are able to reformulate this condition in a more algebraic way:

Definition 19.3. *A subspace $\mathscr{I} \subset \Omega^*(M)$ is an ideal if for any $\omega \in \mathscr{I}$ and $\eta \in \Omega^*(M)$, $\omega \wedge \eta \in \mathscr{I}$. An ideal $\mathscr{I} \subset \Omega^*(M)$ is a differential ideal if*

$$d(\mathscr{I}) \subset \mathscr{I}.$$

Theorem 19.2. *A smooth distribution $\mathscr{D}$ on M is involutive if and only if $\mathscr{I}(\mathscr{D})$ is a differential ideal.*

Proof. Let $\mathscr{D}$ be an involutive distribution. Suppose $\mathscr{D}$ is locally generated by vector fields $X_1,\ldots,X_l$ on an open set U. Complete the collection of vector fields to $X_1,\ldots,X_l,\ldots,X_d$ that form a basis of tangent spaces at all points in U. We take the dual 1-forms $\omega_1,\ldots,\omega_d$ such that

$$\omega_i(X_j)|_U = \delta_{ij}, \quad 1 \le i,j \le d.$$

Then $\mathscr{I}(\mathscr{D})$ is locally generated by the 1-forms $\omega_{l+1},\ldots,\omega_d$. Since for $l+1 \le j \le d$ and $1 \le i, i' \le l$,

$$d\omega_j(X_i, X_{i'}) = X_i\omega_j(X_{i'}) - X_{i'}\omega_j(X_i) - \omega([X_i, X_{i'}]) = 0,$$

we know that $d\omega_j$ annihilates $X_1,\ldots,X_l$.

Conversely, take the 1-forms $\omega_{l+1},\ldots,\omega_d$ that generate $\mathscr{I}(\mathscr{D})$. For $l+1 \le j \le d$, if $d\omega_j$ annihilates $\mathscr{D}$, then

$$\omega([X_i, X_{i'}]) = d\omega_j(X_i, X_{i'}) - X_i\omega_j(X_{i'}) + X_{i'}\omega_j(X_i) = 0.$$

This implies that $[X_i, X_{i'}]$ must be a linear combination of $X_1,\ldots,X_l$. □

Finally, we explain the historical context in which Frobenius' theorem was developed. The classical Frobenius' theorem was a theorem about when a differential

$$b_{11}dx_1 + \cdots + b_{1d}dx_d,\ \ldots,\ b_{c1}dx_1 + \cdots + b_{cd}dx_d$$

can be written as a total differential of smooth functions $\alpha : U \subset \mathbb{R}^d \to V \subset \mathbb{R}^c$. Frobenius proved that there exist smooth functions α that satisfies the equation

$$\frac{\partial \alpha_i}{\partial x_j} = b_{ij}$$

if and only if b_{ij} satisfy the following relation that

$$\frac{\partial b_{i\beta}}{\partial x_\gamma} - \frac{\partial b_{i\gamma}}{\partial x_\beta} + \sum_{j=1}^{n}\Big(b_{j\gamma}\frac{\partial b_{i\beta}}{\partial y_j} - b_{j\beta}\frac{\partial b_{i\gamma}}{\partial y_j}\Big) = 0.$$

Darboux realized that the following formal expression is in fact invariant under coordinate changes

$$\Big(\frac{\partial b_{i\beta}}{\partial x_\gamma} - \frac{\partial b_{i\gamma}}{\partial x_\beta}\Big)(dx_\beta dx_\gamma - dx_\gamma dx_\beta).$$

Cartan finally developed the language of differential forms and exterior differentials under which the above differential can now be written as simply

$$d(b_{11}dx_1 + \cdots + b_{1d}dx_d),\ \ldots,\ d(b_{c1}dx_1 + \cdots + b_{cd}dx_d).$$

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